Chapter 4 - Applications of the Derivative

4.1 Maxima and Minima
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4.5 Linear Approximation and Differentials
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Vocabulary: Chapter 4.1 - Maxima and Minima

1. Absolute Max and Absolute Min value
2. Local Max and Local Min value
3. Extreme Values - Absolute or local
4. Critical points

Absolute maximum:
- No greater value of $f$ nearby.
Also a local maximum.

Absolute minimum:
- No smaller value of $f$ nearby.

Local maximum:
- No smaller value of $f$ nearby.

Derivative is zero = slope is zero or undefined.

\[
\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}
\]

\[
\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}
\]
Chapter 4.1 - Maxima and Minima

**Definition:** Absolute Maximum and Minimum

Let $f(x)$ be defined on an interval $I$ containing $x = c$.

a) If $f(c) \geq f(x)$ for every $x$ in $I$, then

$f(x)$ has an **absolute maximum** value of $f(c)$ on $I$ at $x = c$.

b) If $f(c) \leq f(x)$ for every $x$ in $I$, then

$f(x)$ has an **absolute minimum** value of $f(c)$ on $I$ at $x = c$. 
Chapter 4.1  - Maxima and Minima

Find Absolute Values for different interval types: closed, open, half-open/closed. 

\[ [a, b], (a, b), [a, b), (a, b] \]

a) The graph can be used initially.

b) We solve for it....stay tuned!

Let's look at graphs first:

\[ f(10^{12}) > f(x) \]
\[ f(10^{11}) \quad x \in (-\infty, \infty) \]

\[ f(0) \leq f(x) \quad \forall x \in (-\infty, \infty) \quad [0, 2] \]
\[ 0 \leq f(x) \]

\[ y = x^2 \text{ on } (-\infty, \infty) \]

\[ y = x^2 \text{ on } [0, 2] \]

Closed absolute max of 4 at \( x = 2 \)

Absolute min of 0 at \( x = 0 \)

No absolute max
Chapter 4.1  - Maxima and Minima

Find Absolute Values for different interval types: closed, open, half-open/closed.

\[ f(c) \leq f(x) \quad \forall x \in (0,2] \]

\[ y = x^2 \text{ on } (0,2] \]

\[ y = x^2 \text{ on } (0,2) \]
Chapter 4.1 - Maxima and Minima

Theorem 4.1 Extreme Value Theorem
A function that is continuous on a closed interval \([a,b]\) has an absolute maximum value, \(f(c)\) AND an absolute minimum value \(f(d)\) where \(x = d\) and \(x = c\) are in interval \([a,b]\).

\[
\begin{align*}
\text{abs min is 0 occurs at } & x=0 \\
\text{abs max of y occurs at } & x=2
\end{align*}
\]
Chapter 4.1 - Maxima and Minima

Theorem 4.1 Extreme Value Theorem
A function that is continuous on a closed interval \([a, b]\) has an absolute maximum value, \(f(c)\) AND an absolute minimum value \(f(d)\) where \(x = d\) and \(x = c\) are in interval \([a, b]\).

Exercise: Pick any closed interval then pick other interval types to see that the theorem doesn't guarantee an absolute max or min.
Chapter 4.1 - Maxima and Minima

Theorem 4.1 Extreme Value Theorem
A function that is continuous on a closed interval \([a,b]\) has an absolute maximum value, \(f(c)\) AND an absolute minimum value \(f(d)\) where \(x = d\) and \(x = c\) are in interval \([a,b]\).

\[
\lim_{x \to 0^+} x = 0 \quad \text{and} \quad f(0) = 0
\]
\[
\lim_{x \to 1^-} x = 1 \quad \text{and} \quad f(1) = 0
\]

Exercise: Try to find an absolute max or min on \([a,b]\)

\(\text{abs min of } 0 \text{ at } x = 0, 1\)
\(\text{abs max nonv}\)
Chapter 4.1 - Maxima and Minima

Definition: Local Maximum and Minimum Values

Suppose $f(x)$ is defined on an interval $I$ and $x=c$ is an *interior* point of the interval.

a) If $f(c) \geq f(x)$ for all $x$ 'near $x=c'$ then $f(c)$ is a **local maximum** value of $f(x)$.

b) If $f(c) \leq f(x)$ for all $x$ 'near $x=c'$ then $f(c)$ is a **local minimum** value of $f(x)$.
Chapter 4.1 - Maxima and Minima

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Suppose $f(x)$ is defined on an interval $I$ and $x=c$ is an interior point of the interval.

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b) If $f(c) \leq f(x)$ for all $x$ 'near $x=c$' then $f(c)$ is a local minimum value of $f(x)$. 

$[a, b]$
Chapter 4.1 - Maxima and Minima

**Definition:** Critical Point.

An *interior* point \( x = c \) of the domain of \( f(x) \) for which \( f'(c) = 0 \) or \( f'(c) \) does not exist is called a *critical point*.

**Theorem 4.2** Local Extreme Point theorem

If \( f(x) \) has a local maximum or minimum value at \( x = c \) and if \( f'(x) \) exists then \( f'(c) = 0 \)

\[ f'(c) \geq f'(x) \quad \forall x \]
\[ f'(c) < f'(x) \]

So, local maximum or minimum MAY occur at critical points of \( f(x) \).
Exercise: Locate the critical points. These are possible local max and/or min.

1. $f(x) = 3x^2 - 4x + 2$

   $\frac{f''(x)}{f'(x)} = 0$ or undefined

   $f''(x) = 6x - 4 = 0$

   $x_c = \frac{2}{3}$

   Possible max or min.
Chapter 4.1 - Maxima and Minima

Exercise: Locate the critical points. These are possible local max and/or min.

2. \( f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2 + 10 \) over \([-4,4]\)

\[
f'(x) = \frac{4x^3}{4} - \frac{x^2}{3} - 6x = 0
\]

\[
= x^3 - \frac{x^2}{3} - 6x = 0
\]

\[
= x(x^2 - x - 6) = 0
\]

\[
x(x-3)(x+2) = 0
\]

\[
x = 0, \ x = 3, \ x = -2
\]
Chapter 4.1 - Maxima and Minima

Exercise: Locate the critical points. These are possible local max and/or min.

3. \( f(x) = 2\sqrt{x} - x \) over \([0, 4]\)
Chapter 4.1 - Maxima and Minima

Exercise: Locate the critical points. These are possible local max and/or min.

4. $f(x) = x^{2/3}$
Chapter 4.1 - Maxima and Minima

**Exercise**: Locate the critical points. These are possible local max and/or min.

5. $f(x) = (x - 2)^3 - 4$ over $[1, 5]$
Chapter 4.1 - Maxima and Minima


Assume the function \( f(x) \) is continuous on the closed interval \([a, b]\)

1. Find the critical points. These are possible extreme values
2. Evaluate \( f(x) \) at the critical points and the endpoints of \([a, b]\)
3. Choose the largest value of \( f(a), f(b), f(c) \) for the absolute max. and the smallest value of \( f(a), f(b), f(c) \) for the absolute min.

1. \( f(x) = x^4 \) over \([-1, 6]\)
Chapter 4.1 - Maxima and Minima

**Procedure:** Finding Absolute Maximum and Minimum Values.

Assume the function $f(x)$ is continuous on the closed interval $[a,b]$
1. Find the critical points. These are possible extreme values
2. Evaluate $f(x)$ at the critical points and the endpoints of $[a,b]$
3. Choose the largest value of $f(a)$, $f(b)$, $f(c)$ for the absolute max. and the smallest value of $f(a)$, $f(b)$, $f(c)$ for the absolute min.

2. $f(x) = 2\sqrt{x} - x$ over $[0,4]$
Chapter 4.1 - Maxima and Minima


Assume the function $f(x)$ is continuous on the closed interval $[a,b]$
1. Find the critical points. These are possible extreme values
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Chapter 4.1 - Maxima and Minima


Assume the function $f(x)$ is continuous on the closed interval $[a,b]$
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2. Evaluate $f(x)$ at the critical points and the endpoints of $[a,b]$
3. Choose the largest value of $f(a)$, $f(b)$, $f(c)$ for the absolute max. and the smallest value of $f(a)$, $f(b)$, $f(c)$ for the absolute min.

4. $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 3x^2 + 10$ over $[-4,4]$
Chapter 4.1 - Maxima and Minima


Assume the function \( f(x) \) is continuous on the closed interval \([a,b]\)

1. Find the critical points. These are possible extreme values
2. Evaluate \( f(x) \) at the critical points and the endpoints of \([a,b]\)
3. Choose the largest value of \( f(a), f(b), f(c) \) for the absolute max. and the smallest value of \( f(a), f(b), f(c) \) for the absolute min.

5. \( f(x) = (x^2 - 1)^{\frac{3}{2}} \) over \([-3, 2]\)

6. \( f(x) = x^3 - x \) over \([0, 2]\)

7. \( f(x) = \sqrt{2} \theta - \sec(\theta) \) over \(\left[0, \frac{\pi}{3}\right]\)

8. \( f(x) = (2 + x)\sqrt{2 + (2 - x)^2} \) over \([0, 2]\)

9. \( f(x) = |x^2 + 4x - 12| \) over \([-8, 3]\)

10. \( f(x) = 2\sin(2\theta) + \sin(4\theta) \) over \([0, 2\pi]\)

11. \( f(x) = |\cos \theta| \) over \([0, 3]\)

12. \( f(x) = \frac{x^2 + 1}{x - 4} \) over \([5, 6]\)
Chapter 4.1 - Maxima and Minima

**Example:** Finding Absolute Maximum and Minimum Values.

Suppose a tour guide has a bus that holds a maximum of 100 people. Assume his profit (in dollars) for taking 'n' people on a city tour is \( P(n) = n (50 - 0.5n) - 100 \).

a) How many people should the guide take on a tour to maximize profit?

b) Suppose the bus holds a maximum of 45 people. How many people should be taken on a tour to maximize the profit?
Chapter 4.1 - Maxima and Minima

**Example:** Finding Absolute Maximum and Minimum Values.

Migrating fish tend to swim at a velocity \( v \) that minimizes the total expenditure of energy \( E \). According to one model, \( E \) is proportional to:

\[
E \propto f(v) = \frac{v^3}{v - v_r}
\]

where \( v_r \) is the velocity of the river water.

a) Find and interpret the critical points of \( f(v) \).

b) Choose a value of \( v_r \) (say 10) and plot \( f(v) \). Confirm that \( f(v) \) has a minimum value at the critical point.
Chapter 4.1 - Maxima and Minima

**Example:** Use implicit differentiation to find the critical points on the curve then draw the horizontal tangent lines at the critical points on the curve.

\[ 27x^2 = (x^2 + y^2)^3 \]
Chapter 4.2  What the Derivative Tells Us

1. Intervals on which the function \( f(x) \) is increasing or decreasing.

**Definition:** Increasing and Decreasing Functions

Suppose \( f(x) \) is defined on an interval \( I \). We say that \( f(x) \) is **increasing** on \( I \) if \( f(x_2) > f(x_1) \) whenever \( x_2 > x_1 \) on \( I \). We say that \( f(x) \) is **decreasing** on \( I \) if \( f(x_2) < f(x_1) \) whenever \( x_2 > x_1 \) on \( I \).
**Chapter 4.2 What the Derivative Tells Us**

1. **Intervals on which the function $f(x)$ is increasing or decreasing.**

   **Theorem 4.3 Test for Intervals of Increase and Decrease**

   Suppose $f(x)$ is continuous on an interval $I$ and differentiable at all interior points of $I$.
   
   If $f'(x) > 0$ at all interior points of $I$, then $f$ is **increasing** on $I$.
   
   If $f'(x) < 0$ at all interior points of $I$, then $f$ is **decreasing** on $I$. 

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![Diagram showing increasing and decreasing intervals](image)
Chapter 4.2  What the Derivative Tells Us

1. Intervals on which the function $f(x)$ is increasing or decreasing.

**Exercise:** Find intervals of $f(x)$ increasing and/or decreasing.

a) Find the critical points. The critical points separate the interval into regions of either increasing or decreasing.

b) Evaluate the derivative $f'(x)$ in each region by selecting a test point in that region.
Chapter 4.2 What the Derivative Tells Us

**Exercise:** Find intervals of $f(x)$ increasing and/or decreasing.

a) Find the critical points. The critical points separate the interval into regions of either increasing or decreasing.

b) Evaluate the derivative $f'(x)$ in each region by selecting a test point in that region.

5. $f(x) = (x^2 - 1)^{\frac{3}{2}}$ over $\mathbb{R}$  
9. $f(x) = |x^2 + 4x - 12|$ over $\mathbb{R}$

6. $f(x) = x^3 - x$ over $\mathbb{R}$  
12. $f(x) = \frac{x^2 + 1}{x - 4}$ over $\mathbb{R}$
Chapter 4.2  What the Derivative Tells Us

**Exercise:** Find intervals of \( f(x) \) increasing and/or decreasing.

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5. \( f(x) = (x^2 - 1)^{\frac{3}{2}} \) over \( \mathbb{R} \)
Chapter 4.2  What the Derivative Tells Us

Exercise: Find intervals of \( f(x) \) increasing and/or decreasing.

a) Find the critical points. The critical points separate the interval into regions of either increasing or decreasing.

b) Evaluate the derivative \( f'(x) \) in each region by selecting a test point in that region.

6. \( f'(x) = x^5 - x \) over \( \mathbb{R} \)

Since it's a fifth degree polynomial, how do you know there aren't more roots?
Chapter 4.2 What the Derivative Tells Us

Exercise: Find intervals of $f(x)$ increasing and/or decreasing.

a) Find the critical points. The critical points separate the interval into regions of either increasing or decreasing.

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Chapter 4.2  What the Derivative Tells Us

**Exercise:** Find intervals of $f(x)$ increasing and/or decreasing.

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12. $f(x) = \frac{x^2 + 1}{x - 4}$ over $\mathbb{R}$
Chapter 4.2  What the Derivative Tells Us

2. Determine if the critical point is a local maximum or local minimum.

Theorem 4.4  First Derivative Test (find extreme values)

Suppose \( f(x) \) is continuous on an interval that contains a critical point \( x = c \) and assume that \( f(x) \) is differentiable on an interval containing \( c \), except possibly at \( c \) itself.

a) If \( f'(x) \) changes sign from positive to negative as \( x \) increases through \( x = c \), then \( f(x) \) has a **local maximum** at \( x = c \).

b) If \( f'(x) \) changes sign from negative to positive as \( x \) increases through \( x = c \), then \( f(x) \) has a **local minimum** at \( x = c \).

c) If \( f'(x) \) does NOT change sign as \( x \) increases through \( x = c \), then \( f(x) \) has a **NO local extreme value** at \( x = c \).
Chapter 4.2  What the Derivative Tells Us

2. Determine if the critical point is a local maximum or local minimum. 

Theorem 4.4  First Derivative Test (find extreme values) 

a) If \( f'(x) \) changes sign from positive to negative as \( x \) increases through \( x = c \), then \( f(x) \) has a **local maximum** at \( x = c \). 

b) If \( f'(x) \) changes sign from negative to positive as \( x \) increases through \( x = c \), then \( f(x) \) has a **local minimum** at \( x = c \). 

c) If \( f'(x) \) does NOT change sign as \( x \) increases through \( x = c \), then \( f(x) \) has a **NO local extreme value** at \( x = c \).
Chapter 4.2  What the Derivative Tells Us

**Exercise:** Determine if the critical point is a local max or min by using the 1st derivative test.

5. \( f(x) = (x^2 - 1)^{\frac{1}{3}} \) over \( \mathbb{R} \)
Chapter 4.2  What the Derivative Tells Us

Exercise: Determine if the critical point is a local max or min by using the 1st derivative test.

6. \( f(x) = x^3 - x \) over \( \mathbb{R} \)
Chapter 4.2  What the Derivative Tells Us

**Exercise:** Determine if the critical point is a local max or min by using the 1st derivative test.

9. $f(x) = |x^2 + 4x - 12|$ over $\mathbb{R}$
Chapter 4.2 What the Derivative Tells Us

Exercise: Determine if the critical point is a local max or min by using the 1st derivative test.

12. \( f(x) = \frac{x^2+1}{x-4} \) over \( \mathbb{R} \)

\[ f'(x) = \frac{2x(x-4)-(x^2+1)}{(x-4)^2} \cdot 1 \]

\[ f''(x) = \frac{x^2-8x-1}{(x-4)^2} \]

\[ x_c = \frac{8 \pm \sqrt{64-4(-1)}}{2} = \frac{8 \pm \sqrt{68}}{2} = 4 \pm \sqrt{17} \]

1st derivative test.

\( \begin{array}{c|c|c|c|}
\hline
-1 & \text{max} & 0 & \text{min} & 10 \\
\hline
f' & \leftrightarrow & & & \\
\hline
(-\infty, 4-\sqrt{17}) & (4-\sqrt{17}, 4+\sqrt{17}) & (4+\sqrt{17}, \infty) & \\
\hline
f''(-1) = \frac{1+8-1}{25} = \frac{8}{25} & & & \\
\hline
f'(10) = \frac{100-80-1}{6^2} = \frac{1}{3} & & & \\
\hline
\end{array} \]
\[ f(x) = \frac{1}{x^2} = x^{-2} \]

\[ f'(x) = -2x^{-3} = -\frac{2}{x^3} \]

\[ f''(x) = 0 \text{ no sol.} \]

\[ f'(x) \text{ und.} \]
3. Concavity and Inflection points

**Definition Concavity and inflection point**

Let $f(x)$ be differentiable on an open interval $I$. If $f'(x)$ is increasing on $I$ then $f(x)$ is **concave up** on $I$. If $f'(x)$ is decreasing on $I$, then $f(x)$ is **concave down** on $I$.

If $f(x)$ is continuous at $x=c$ and $f$ changes concavity at $x=c$ then $f(x)$ has an **inflection point** at $x=c$. 
Chapter 4.2  What the Derivative Tells Us

3. Concavity and Inflection points

**Theorem** Test for concavity

Suppose $f''(x)$ exists on an interval $I$.

If $f''(x) > 0$ on $I$ then $f(x)$ is concave up on $I$.

If $f''(x) < 0$ on $I$ then $f(x)$ is concave down on $I$.

If $c$ is a point of $I$ at which $f''(x)$ changes sign at $c$, the $f(x)$ has an inflection point. If $f(x)$ has an inflection point then $f''(x) = 0$ or $f''(x)$ is undefined.
Chapter 4.2 What the Derivative Tells Us

3. Concavity and Inflection points

Exercise Determine the intervals on which the function is concave up or concave down.

6. $f(x) = x^5 - x$ over $\mathbb{R}$

$$f'(x) = 5x^4 - 1$$

$$f''(x) = 20x^3 = 0 \quad x_1 = 0$$

$$f''(-2) = -160 \quad x_2 = 0 \quad f''(2) = 160$$
Chapter 4.2  What the Derivative Tells Us

3. Concavity and Inflection points

**Exercise** Determine the intervals on which the function is concave up or concave down.

12. \( f(x) = \frac{x^2 + 1}{x - 4} \) over \( \mathbb{R} \)
Chapter 4.2  What the Derivative Tells Us

3. Concavity and Inflection points

**Exercise** Determine the intervals on which the function is concave up or concave down.

1. \( f(x) = x^{\frac{3}{2}} \) over \([-1, 6]\)

\[
\begin{align*}
\frac{d}{dx} f(x) &= \frac{2}{3} x^{-\frac{1}{3}} \\
\frac{d^2}{dx^2} f(x) &= -\frac{2}{9} x^{-\frac{4}{3}} \\
\end{align*}
\]

\( \frac{d^2}{dx^2} f(x) = \frac{2}{9} x^{-\frac{4}{3}} = 0 \) no solution

\( f''(-3) = \Theta \) \( \wedge \) \( \wedge \) \( \text{undefined at } x=0 \)

\( f''(3) = \Theta \) \( \text{not an inflection point} \)
Chapter 4.2 What the Derivative Tells Us

3. Concavity and Inflection points

Theorem 4.7 Second Derivative Test (find extreme values)

Suppose $f''(x)$ is continuous on an open interval that contains $c$ with $f'(c)=0$

a) If $f''(c) < 0$ then $f(x)$ has a **local maximum** at $x=c$.

b) If $f''(c) > 0$ then $f(x)$ has a **local minimum** at $x=c$.

c) If $f''(c) =0$ then the test is inconclusive, meaning $x = c$ may be a local max, local min, or neither.

Example: $f(x) = -x^4(x-3) = -x^4 + 3x^3$

\[f'(x) = -4x^3 + 9x^2 = 0\]

\[x^2(-4x + 9) = 0\]

\[x = 0 ~ \text{or} ~ x = \frac{9}{4}\]

Plug into $f''(x)$

\[f''(x) = -12x^2 + 18x\]

\[f''(0) = 0 \quad \text{and} \quad f''\left(\frac{9}{4}\right) = -12\left(\frac{9}{4}\right)^2 + 18\left(\frac{9}{4}\right) = -20.25\]

The test is inconclusive at $x = 0$.

We then used the 1st derivative test which showed $x = 0$ is neither a max or a min because $f(x)$ continued to increase through $x=0$.

1st derivative test:

\[x_c = 0 \quad x_c = \frac{9}{4}\]

\[f'(x) \quad -1 \quad 0 \quad 2 \quad \text{max} \quad 10\]

\[f'(x) \quad -4x^3 + 9x^2\]

\[f' > 0 \quad f' = 0 \quad f' > 0 \quad f' < 0\]

\[f'' \quad f'' \quad f'' \quad f''\]
Summary

\[ f''(x) > 0 \Rightarrow f'(x) \uparrow \]
\[ f''(x) < 0 \Rightarrow f'(x) \downarrow \]

\[ f'''(x) > 0 \Rightarrow f''(x) \uparrow \Rightarrow f'(x) \text{ is concave up} \]
\[ f'''(x) < 0 \Rightarrow f''(x) \downarrow \Rightarrow f'(x) \text{ is concave down} \]

How can we determine if critical points are local max, min or neither?

1. First derivative test
2. Second derivative test