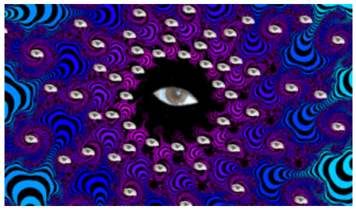
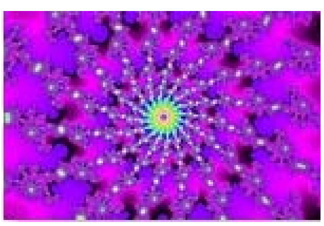
Mathematics as Art by Rafael Espericueta

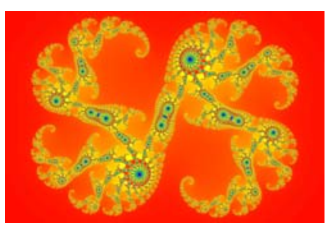
Mathematics can seem cold and abstruse to the uninitiated, yet profoundly beautiful for those with eyes to see. But one must first forge those "eyes to see". For studying a branch of mathematics is akin to growing an abstract organ of perception, through which one may perceive mysteries otherwise inconceivable.

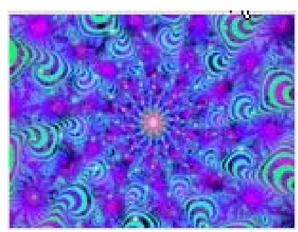


Imagine a world of the deaf, in which you could develop the ability to hear music – but only music that you yourself could actually play on an instrument. As your musical skill increased, so would your very ability to hear music. In such a world music would seem much as math does to far too many people. Having been exposed to little musical training, most in our deaf world would perhaps think that musical scales were all there was to music – how dull!

Luckily there are techniques to literally make visible the beauty of mathematics. Our visual system comprises about a third of our total brain power. Making mathematics visible utilizes this significant processing power. And interesting mathematics often gives rise to beautiful and intriguing visual forms.

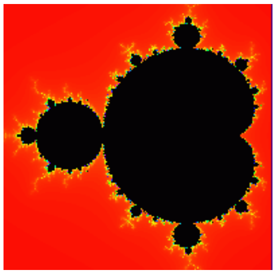




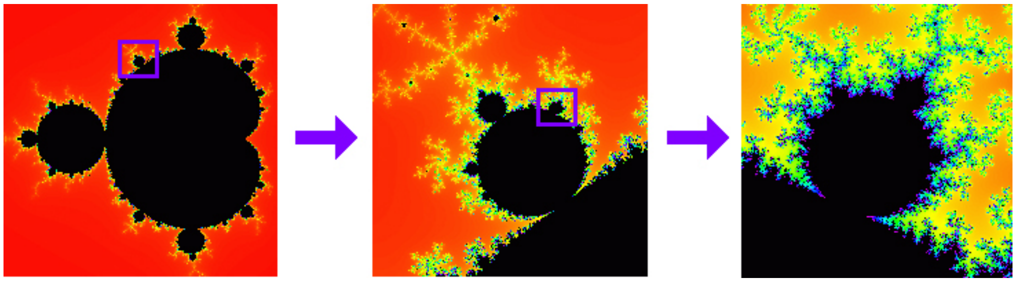


Mathematics just has to be the ultimate abstract art form! While a math undergraduate, I studied some of the writings of early 20^{th} century abstract artists, including Kandinsky's "On the Spiritual in Art". These early abstract artists were intrigued by the advancements of physics, in both quantum and relativity theories. Reality was not what it seemed. What's the point of merely depicting the way things appear, when one has realized that the appearance is but an illusion? The advances in physics made use of increasingly abstract mathematics. The early abstract artists were attempting to do for art what mathematical abstraction had done for physics, to penetrate to a deeper reality underlying the surface appearance of things, and to propel art into the 20^{th} century. I was struck with the realization that these early abstract artists vision of art was uncannily close to my vision of mathematics.

In 1979 the mathematician Benoit Mandelbrot was the first to create pictures that illustrate the dynamics of functions with domain and range in the complex plane. His very first plot was so complicated that he thought it due to a bug in his program. The function he was studying was just too simple to give rise to such a complex pattern. The fractal he created now bears his name, the Mandelbrot set:



The Mandelbrot set has the remarkable property that distorted copies of the whole set appear at all levels of magnification.



This type of infinite complexity is a characteristic of mathematical objects known as fractals. The irregularities often seem almost organic, and indeed, much of the physical universe has approximate fractal characteristics (like coastlines, mountains, clouds, nervous systems, circulatory systems, etc.). Mandelbrot and many others have applied fractal techniques to many real-world phenomena, with great success.

One of the most remarkable properties of the Mandelbrot set, and indeed fractals in general, is that despite their infinite complexity, they are created using very simple algebraic formulas using recursion.

Recursion is a feedback process, whereby the output of a function is used as the next input to the same function. For the Mandelbrot set, the function used is simply $f(z) = z^2 + C$. It may seem impossible that such a simple formula could produce such an infinitely complex creature as the Mandelbrot set. Recursion is what makes this possible. Consider the first few iterates of this recursion process for this function:

$$f(z) = z^2 + C$$

$$f(f(z)) = (z^2 + C)^2 + C$$

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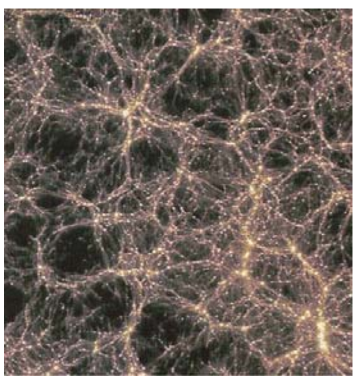
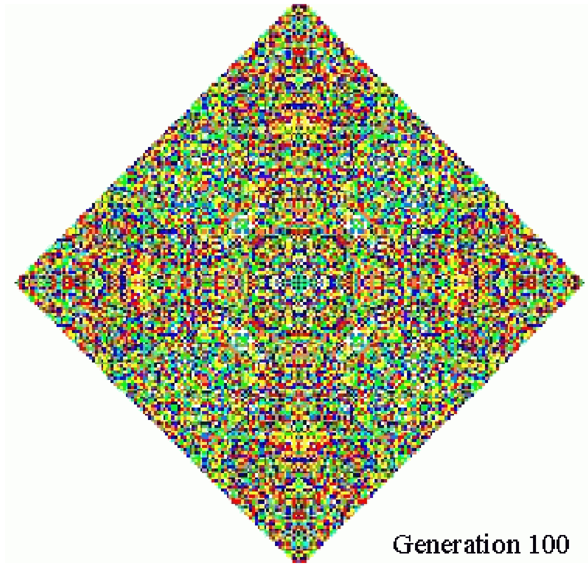
The complexity of fractals arises from this type of feedback.   Indeed, the very formula itself becomes a kind of fractal as we continue iterating it!



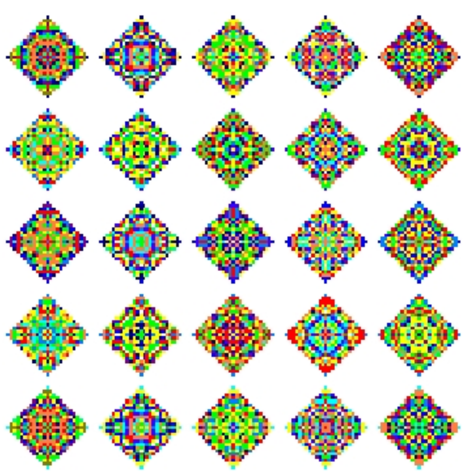
The whole universe operates in much the same way, where the "simple algebraic formulas" are the laws of physics, for which each moments outputs are the inputs for the next moment. We live in a recursive universe, so it's perhaps not so surprising that we find ourselves living in a giant fractal, a fractal of which we are a part. The Hermetic axiom, "As is above, so tis below", thus perhaps has a mathematical basis. As with many fractals, the whole of the universe can be encoded into each of its tiniest parts.   (Holograms also possess this fractal property.)

Fractals aren't the only mathematical creatures that arise by recursion using simple rules. Cellular automata have the property that simple local rules of interaction can give rise to complex global structure. During a sabbatical some years ago I explored the possibilities of translating cellular automata states into visual patterns. What I found was that simple local rules (i.e. physics) and a simple initial state could give rise to complex emergent behavior, evolving infinite complexity.   For example, after 100 iterations of a simple local rule, starting with a single non-zero cell, one cellular automata evolved to this:



Notice the rich symmetries and patterns that have evolved. Each generation this pattern expands a little as the pattern evolves. Below is a detail from the 250th generation:

It may look like someone carefully crafted these mandala-like symmetries, but they spontaneously evolved using very simple local interaction rules. Some physicists think the ultimate model of our universe itself may be a form of cellular automata. In the following image, you can see what several different seeds have grown to after 12 generations:

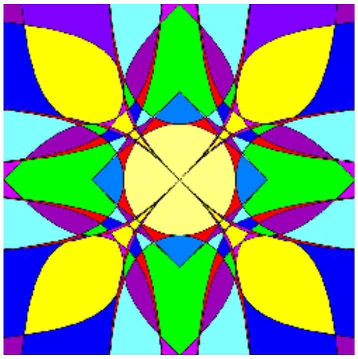


We humans often perceive symmetry as beautiful. Given any function f (from the real numbers to the real numbers) defined in the first quadrant, one can create a four-fold symmetry by plotting on the same graph the four equations:

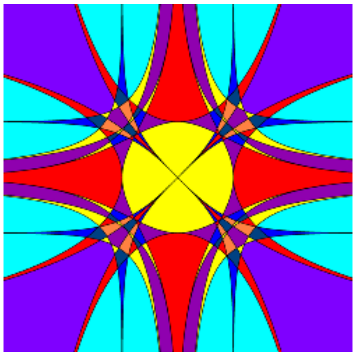
$$y = f(x), \quad y = -f(x), \quad y = f(-x), \quad \text{and} \quad y = -f(-x)$$

The first yields the graph of the function, the second its reflection through the *x*-axis, the third its reflection through the *y*-axis, and the fourth its reflection through the origin.

I used the six trigonometric functions (sine, cosine, tangent, cotangent, secant, and cosecant) along with their four-fold symmetries described above, to create the following mandala, which I call the *Trigonometric Flower of Life*:



Similarly, I used the six hyperbolic functions (hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, and hyperbolic cosecant) along with their four-fold symmetries, to create the following piece, the *Hyperbolic Flower of Life*:

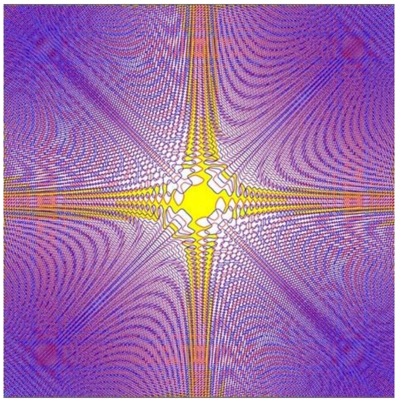


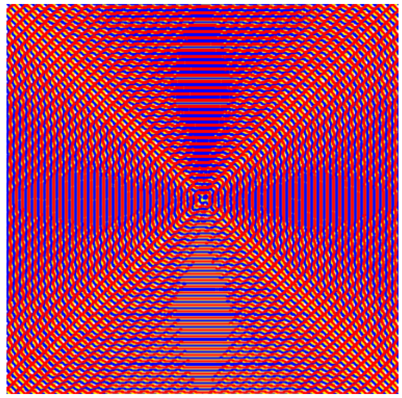
Implicitly defined curves have been studied and applied for centuries, and yet only with the advent of computers has it been possible to visualize many of these curves. In the following image on the left is graphed

$\sin(xy) + \sin(x^2) + \sin(y^2) = 0$ in blue, along with the graph of

$\cos(xy) + \cos(x^2) + \cos(y^2) = 0$ in red,

with a “flood-fill” of yellow in the center (just for fun!):





To the right above are the graphs implicitly determined by the equations:

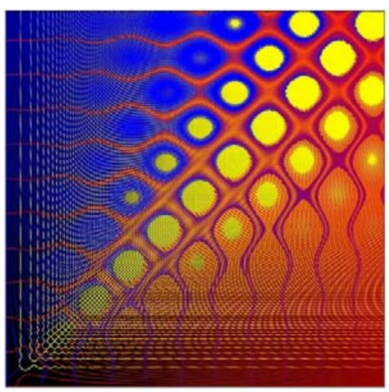
$$x\sin x + y\sin y - \sin(xy) = 0 \quad \text{in red,}$$
$$x\sin x + y\cos y = 0 \quad \text{in yellow, and}$$
$$x\cos x + y\cos y - \cos(xy) = 0 \quad \text{in blue:}$$

Once again, great complexity can lurk behind deceptively simple equations.

The following pattern consists of the implicit graphs of

$$x^2\sin(y^2) + y^2\cos(x^2) = 0 \quad \text{in yellow,}$$
$$xy\sin x + y^2\sin y - x^2\sin(xy) = 0 \quad \text{in red, and}$$
$$xy\sin y + x^2\sin x - y^2\sin(xy) = 0 \quad \text{in blue:}$$

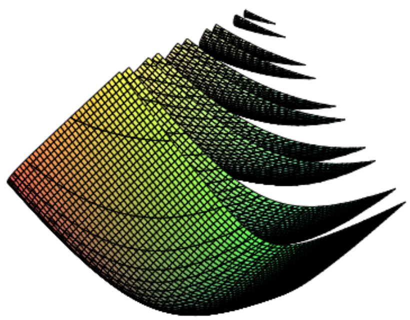
along with their reflections through the *y*-axis:



Often one finds that symmetries in the defining equations give rise to pleasing visual symmetries in the graphs of the equations.

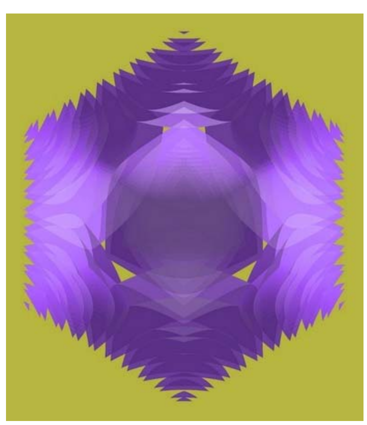
Just as implicit equations with two variables have visually interesting graphs, so do implicit equations with three variables. But in this case, rather than plane curves we obtain surfaces in 3-dimensional space. These are best visualized by simulating their rotation, since our visual systems are good at processing information received in that way. After all, survival in our apparent 3-dimensional environment is enhanced when our brains correctly interpret the ever changing 2-dimensional images on our retinas as projections of 3D objects moving in 3D space. However, sometimes capturing the right virtual illumination of the surface at the right angle can yield visually interesting images:

The following is a graph of part of the surface defined implicitly by the equation $\sin(xyz) = \frac{1}{2}$.

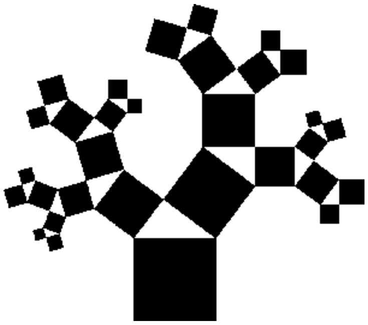


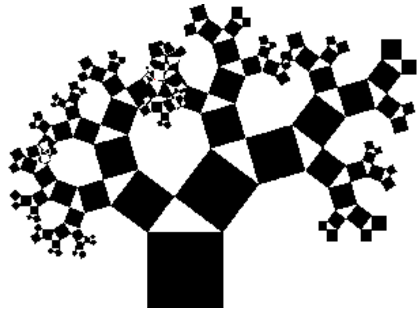
The next image is based on another, more encompassing view of the graph of $\sin(xyz) = \frac{1}{2}$.

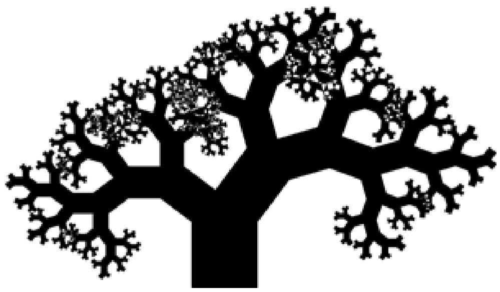
Notice how a different rendering of the surface, and a different angle, can yield an entirely different visual effect:



The following diagrams are called Pythagoras trees, and though constructed entirely of squares, end up looking quite organic and tree like. Notice the right triangles where groups of 3 squares come together. These all have the ratios of the famous Egyptian 3:4:5 right triangle. At each bifurcation the branching is randomly selected as either 3:4 or 4:3.


5 levels of recursion


7 levels of recursion


10 levels of recursion

In that last image, the triangles were filled in to make a more realistic looking tree. But it too is composed entirely of squares and triangles.

The reader may not yet be convinced that mathematics is the *ultimate* abstract art. Hopefully the reader now at least finds plausible the notion that mathematics can provide a treasure trove of beautiful patterns that artists can draw from to create interesting pieces. And hopefully, gentle reader, you'll now have eyes to see a bit more of the mystery and beauty that is mathematics!