10.1 Sequences

For us, a *sequence* is a succession of real numbers. The sequence of non-negative integers is the sequence \( \{0, 1, 2, 3, \ldots \} \). This could also be specified as \( \{a_n\} \), where \( a_n = n \). This notation is a kind of function notation, where the domain of the function consists of the whole numbers \( \{0, 1, 2, 3, \ldots \} \).

Instead of writing \( a(n) = n \), we write this function using a subscript, \( a_n = n \). Some texts start off all sequences at \( n = 1 \), but it’s often more convenient to start them off at \( n = 0 \); this will be our default, starting off sequences with a zero\(^{th}\) term, unless otherwise specified.

Example 1. Find a formula for the \( n^{th} \) term of the sequence

\[
\begin{align*}
1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots
\end{align*}
\]

starting with \( n = 0 \).

Each term is half of the previous term. Thus \( a_0 = 1 = \frac{1}{2^0}, \quad a_1 = \frac{1}{2} = \frac{1}{2^1}, \quad a_2 = \frac{1}{4} = \frac{1}{2^2}, \quad a_3 = \frac{1}{8} = \frac{1}{2^3}, \ldots \)

so evidently the \( n^{th} \) term of the sequence is \( a_n = \frac{1}{2^n} \), for \( n \geq 0 \).

Example 2. Find a formula for \( b_n \) the \( n^{th} \) term of the sequence of odd counting numbers

\( \{1, 3, 5, 7, 9, \ldots \} \), starting with \( n = 0 \). The even integers can each be written as 2 times something, and adding or subtracting 1 from an even integer results in an odd one. Thus, \( b_n = 2n + 1 \), for \( n \geq 0 \).

In this course our primary interest in sequences concerns their convergence or divergence. We say that a sequence \( \{a_n\} \) converges to \( L \), provided that \( \lim_{n \to \infty} a_n = L \). The formal definition of this limit is similar to the limit definition you encountered before, in Calculus I. And as in first semester calculus, in this course we are more concerned with an intuitive grasp of the limit concept than with a working knowledge of its formal definition. Intuitively a sequence \( \{a_n\} \) converges to \( L \), if the elements of the sequence get closer to \( L \), the larger \( n \) becomes. Formally, \( \lim_{n \to \infty} a_n = L \) means that

for any \( \varepsilon > 0 \) (no matter how small),

there exists an integer \( N \) such that for all \( n \geq N \),

\[ |L - a_n| < \varepsilon. \]

No matter how small a number \( \varepsilon > 0 \) you specify, if one goes far enough down the sequence, all the terms of the sequence will be within \( \varepsilon \) of \( L \).
Example 3. The sequence from Example 1, \( a_n = \frac{1}{2^n} \), converges to 0.

No matter how small a number \( \varepsilon > 0 \) one picks, one can find an \( N \) large enough that \( \varepsilon > \frac{1}{2^N} \).

For example, suppose that \( \varepsilon = 0.000001 \). Then we must find an \( N \) that satisfies the inequality:

\[
0.000001 > \frac{1}{2^N}
\]

Let’s solve that inequality:

\[
0.000001 > \frac{1}{2^N} \quad \Rightarrow \quad (0.000001)2^N > 1 \quad \Rightarrow \quad 2^N > \frac{1}{0.000001} = 1000000
\]

\[
\Rightarrow \ln\left(2^N\right) > \ln1000000 \quad \Rightarrow \quad N \ln2 > \ln1000000
\]

\[
\Rightarrow \quad N > \frac{\ln1000000}{\ln2} = 19.931568569324174087
\]

\( N \) has to be an integer, so if we let \( N = 20 \) then for all \( n \geq 20 \) it will be true that \( 0.000001 > \frac{1}{2^n} \).

No matter how small \( \varepsilon > 0 \) may be, one can go through the above process to find an \( N \) large enough that for all \( n \geq N \) it will be true that \( \varepsilon > \frac{1}{2^N} \). Thus \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^n} = 0 \).

We can multiply a sequence times a real number \( c \), thus obtaining a new sequence. Also we can add two sequences to obtain a new sequence.

Example 4. Find the sum of the sequences from Exercise 1 and Exercise 2. If \( a_n = \frac{1}{2^n} \) and \( b_n = 2n + 1 \), for \( n \geq 0 \), then find the sequence \( \{a_n\} + \{b_n\} = \{a_n + b_n\} = \{c_n\} \). \( c_n = ? \)

\[
c_n = a_n + b_n = \frac{1}{2^n} + 2n + 1 \quad \text{or} \quad \frac{2^{n+1}n + 2^n + 1}{2^n}
\]

The basic properties of limits of sequences are the same as the basic limit properties you’ve previously studied. Suppose that you have two sequences, \( \{a_n\} \) converging to \( A \), and \( \{b_n\} \) converging to \( B \). Let
$c_1$ and $c_2$ be two real constants. Then the following linearity properties can all be readily proven from the formal definition of limit given above:

\[
\lim_{n \to \infty} (ca_n) = c \lim_{n \to \infty} a_n = cA
\]

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = A + B
\]

Also the following properties also follow from our formal definition of limit:

\[
\lim_{n \to \infty} (a_n \cdot b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n = AB
\]

\[
\lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{A}{B}, \text{ assuming no zero denominators.}
\]

\[
\lim_{n \to \infty} (a_n^p) = A^p, \text{ if } p > 0 \text{ and all } a_n > 0.
\]

If a sequence \( \{a_n\} \) keeps growing larger with no bound, we say that \( \lim_{n \to \infty} a_n = \infty \). Formally this means:

for any \( M > 0 \) (no matter how large),
there exists an integer \( N \) such that for all \( n \geq N \),

\[
|L - a_n| > M.
\]

Similarly, if a sequence \( \{a_n\} \) keeps growing smaller with no lower bound, we say that \( \lim_{n \to \infty} a_n = -\infty \). Formally this means:

for any \( M > 0 \) (no matter how large),
there exists an integer \( N \) such that for all \( n \geq N \),

\[
|L - a_n| < -M.
\]

Given a continuous function \( f \), with domain containing the non-negative integers, suppose that a sequence is defined by \( a_n = f(n) \). Suppose further that \( \lim_{x \to \infty} f(x) = L \). Then \( \lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x) = L \).

Example 5. Define a sequence by \( a_n = \tan^{-1} n \). To what value does \( \{a_n\} \) converge?

Since \( \lim_{n \to \infty} \tan^{-1} x = \frac{\pi}{2} \), it follows that \( \lim_{n \to \infty} a_n = \frac{\pi}{2} \).
When a sequence is defined by a continuous function, as in Example 5, the sequence can be thought of as a “sampling” of the function. In the diagram, you can see our sequence \( \{a_n\} \) graphed along with the continuous arc tangent function that defines it. In the graph, the value of each \( a_n \) is the height of the graph of the arc tangent function at \( x = n \).

The graph makes it easy to see why \( \lim_{n \to \infty} a_n = \frac{\pi}{2} \).

The Squeeze Theorem For Limits of Sequences

Suppose that for all \( n \), \( a_n \leq c_n \leq b_n \), and that \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} b_n = L \). Then \( \lim_{n \to \infty} c_n = L \).

\[
\begin{align*}
  & a_n \leq c_n \leq b_n \\
  \quad \downarrow & \quad \downarrow & \quad \downarrow \\
  \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} c_n \leq \lim_{n \to \infty} b_n \\
  \quad \downarrow & \quad \downarrow & \quad \downarrow \\
  L & \leq \lim_{n \to \infty} c_n \leq L 
\end{align*}
\]

Thus \( \lim_{n \to \infty} c_n \) simply has no choice, but to converge to \( L \).

Example 6. Consider the sequence by \( a_n = \frac{\sin n}{n} \). Then \( \lim_{n \to \infty} a_n = ? \)

Since the sine function is always between \(-1\) and \(1\), it follows that \(-1 \leq \sin n \leq 1\), for all \( n \).

Multiplying all three sides of this inequality by \( 1/n \), it becomes \( -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \), for all \( n \).

Since the sequence \( \left\{-\frac{1}{n}\right\} \) converges to 0, and the sequence \( \left\{\frac{1}{n}\right\} \) converges to 0, we have:
\[
\frac{-1}{n} \leq \frac{1}{n} \sin n \leq \frac{1}{n}
\]

\[
\lim_{n \to \infty} \left( -\frac{1}{n} \right) \leq \lim_{n \to \infty} \left( \frac{\sin n}{n} \right) \leq \lim_{n \to \infty} \left( \frac{1}{n} \right)
\]

\[
0 \leq \lim_{n \to \infty} \left( \frac{\sin n}{n} \right) \leq 0 \Rightarrow \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{\sin n}{n} \right) = 0
\]

In the following graph one can see this convergence. The two yellow curves form an envelope around the blue graph of \( y = \frac{\sin x}{x} \), and yellow curves asymptotically approach the x-axis, squeezing the blue graph, including those (red) points of our sequence, to 0.

The following useful theorem is little more than a restatement of “\( f \) is continuous at \( L \)”: Suppose that \( \lim_{n \to \infty} a_n = L \) and that \( f \) is continuous at \( L \). Then \( \lim_{n \to \infty} f(a_n) = f(L) \).

Example 7. If \( a_n = \tan \frac{1}{n} \). Then \( \lim_{n \to \infty} a_n = ? \)

\[
\lim_{n \to \infty} \frac{1}{n} = 0, \text{ and the tangent function is continuous at } 0.
\]

Thus, \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \tan \left( \frac{1}{n} \right) = \tan \left( \lim_{n \to \infty} \frac{1}{n} \right) = \tan 0 = 0 \]

Our last theorem above is not only useful for solving limits such as that of Example 7, where it allows the limit and the function to commute (the limit of the function is the function of the limit). More
importantly, this also justifies using l’Hopital’s rule. Although l’Hopital’s rule applies to continuous functions, we can now apply it to the discrete terms of a sequence.

Example 8. If \( a_n = \frac{\ln n}{\sqrt{n}} \). Then \( \lim_{n \to \infty} a_n = ? \)

\[ a_n = \frac{\ln n}{\sqrt{n}} = f(n), \text{ where } f(x) = \frac{\ln x}{\sqrt{x}}. \] We can apply l’Hopital’s rule to \( f \), and by our last theorem, this must also then apply to the limit of our sequence.

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\ln x}{\frac{1}{x^2}} \]

Since both the numerator and the denominator go to infinity in the limit, we may apply l’Hopital’s rule – the limit we seek is equal to the limit of the derivative of the numerator divided by the derivative of the denominator:

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\frac{d}{dx} \ln x}{\frac{1}{2} x^{-\frac{1}{2}}} = \lim_{x \to \infty} 2 \cdot \frac{1}{x^2} = 2 \cdot \frac{1}{\infty} = 2 \cdot 0 = 0 \]

L’Hopital’s rule is frequently a useful tool for evaluating the limit of a sequence!

Definitions:

A sequence \( \{a_n\} \) is said to be **monotonically increasing**, if for all \( n \) it’s true that \( a_n < a_{n+1} \).

A sequence \( \{a_n\} \) is said to be **monotonically decreasing**, if for all \( n \) it’s true that \( a_n > a_{n+1} \).

A sequence \( \{a_n\} \) is said to be **bounded above** by \( B \), if for all \( n \) it’s true that \( a_n \leq B \).

A sequence \( \{a_n\} \) is said to be **bounded below** by \( B \), if for all \( n \) it’s true that \( a_n \geq B \).

**Monotone Convergence Theorem**: An important theorem, proved in Advanced Calculus, states that every monotonically increasing sequence that’s bounded above is convergent. The limit of the sequence is then the **least upper bound** (the smallest upper bound) of the sequence. Similarly, every monotonically decreasing sequence that’s bounded below is also convergent, and converges to the sequence’s **greatest lower bound** (the largest lower bound) of the sequence.
Example 9. Let \( a_n = r^n \). Then \( \lim_{n \to \infty} a_n = ? \)

This limit depends on the value of \( r \). By examining graphs of \( y = r^x \) for various positive values of \( r \), one sees that \( \lim_{x \to \infty} r^x = \infty \) whenever \( r > 1 \), and that \( \lim_{n \to \infty} r^n = 0 \) whenever \( 0 \leq r < 1 \). The sequences obtained for negative values of \( r \) are the same as those obtained for positive values of \( r \), except that the terms of the sequence then alternate positive, negative, positive, negative, etc... Thus, \( \lim_{n \to \infty} r^n = 0 \), when \( r \) is in the interval \(-1 < r < 1\), i.e. for \( |r| < 1 \), and the sequence diverges for values of \( r \) such \( |r| > 1 \).

When \( r = 1 \), the sequence converges to 1 (since all the terms equal 1). But when \( r = -1 \) the series diverges, since it hops back and forth between \(-1 \) and \( 1 \), and never settles down to converge. For when \(-1 \) is raised to even powers, the result is 1, and when \(-1 \) is raised to odd powers, the result is \(-1 \).

To summarize:

\[
\lim_{n \to \infty} r^n = \begin{cases} 
\infty & \text{for } |r| > 1; \\
1 & \text{for } r = 1; \\
0 & \text{for } |r| < 1; \\
\text{undefined} & \text{for } r = -1.
\end{cases}
\]

This result will be particularly important in the next section, where it will help us to add up all of the infinite terms of the so called geometric series.

Make sure you look at a few examples, to get a good feel for this. You may want to make a few of your own tables to see numerically what’s going on with these sequences. Numerically, one can almost see them converge, or diverge as the case may be. For example, in the following table are a few terms of the sequence where \( a_n = \left( \frac{1}{2} \right)^n \).
One can see the terms of this sequence approaching 0:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\left(\frac{1}{2}\right)^0 = 1$</td>
</tr>
<tr>
<td>1</td>
<td>$\left(\frac{1}{2}\right)^1 = \frac{1}{2} = 0.5$</td>
</tr>
<tr>
<td>2</td>
<td>$\left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$</td>
</tr>
<tr>
<td>3</td>
<td>$\left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125$</td>
</tr>
<tr>
<td>10</td>
<td>$\left(\frac{1}{2}\right)^{10} = \frac{1}{1,024} = 0.0009765625$</td>
</tr>
<tr>
<td>20</td>
<td>$\left(\frac{1}{2}\right)^{20} = \frac{1}{1,048,576} = 0.00000095367431640625$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>$\left(\frac{1}{2}\right)^\infty = 0$</td>
</tr>
</tbody>
</table>

The above should appeal to your “common” sense. Starting with any fixed quantity, one can take half of it, and then half of THAT, and then half of that, and so on, one can eventually reduce the original quantity to be smaller than any positive value you wish.

In the case where $0 < r < 1$, the sequence $a_n = r^n$ is monotonically decreasing, and is bounded below by 0 (as well as any negative number), so by the Monotone Convergence Theorem it converges to it’s greatest lower bound, which is 0. In the case where $1 < r$, the sequence $a_n = r^n$ is monotonically increasing, and is unbounded. It diverges to infinity.
Recursively Defined Sequences

The sequences we’ve seen so far explicitly define the \( n^{th} \) term of the sequence as a function of \( n \). But one may define sequences in terms of themselves, as with the Fibonacci numbers. The first two Fibonacci numbers are 1 and 1. Each successive Fibonacci number is then the sum of the two previous ones:

\[
\begin{align*}
F_0 &= F_1 = 1 \\
F_{n+2} &= F_n + F_{n+1}, \text{ for } n \geq 0
\end{align*}
\]

Thus \( \{F_n\} = \{1,1,2,3,5,8,13,21,34,55,\cdots\} \), a monotone increasing unbounded sequence.

This sequence was studied by a 13th century mathematician, Fibonacci, while working on a problem concerning the growth of a rabbit population.

Often its easier to express a sequence recursively. For example, suppose you invest \( P \) dollars into an account with annual interest rate \( r \), compounded monthly. Let \( A_n \) be the amount in the account after \( n \) months. Then \( A_0 = P \), and \( A_{n+1} = A_n + \frac{r}{12} A_n = \left(1 + \frac{r}{12}\right) A_n \). The sequence \( \{A_n\} \), is thus easily defined recursively:

\[
\begin{align*}
A_0 &= P \\
A_{n+1} &= \left(1 + \frac{r}{12}\right) A_n
\end{align*}
\]

Our earlier studies revealed the “solution” to the above recursively defined compound interest sequence:

\[
A_n = \left(1 + \frac{r}{12}\right)^n P
\]

The process of solving a recursively defined sequence (also called a difference equation) is beyond the scope of this course. But it turns out to give rise to a whole calculus of sequences, which parallels all of calculus through differential equations. It’s the discrete form of our continuous calculus, with its own product rule, quotient rule, etc. The “derivative” of a sequence is just the sequence of differences between successive terms of the sequence. The “integral” of a sequence is obtained by adding up the terms of a sequence.
The field of difference equations is very interesting and useful, especially in this age of digital computers. For digital computers are intrinsically discrete. Often differential equations are transformed into difference equations in order to obtain numerical solutions. Indeed, it sometimes gets a bit strange, when real-world phenomena that are intrinsically discrete (like a population of animals – it’s always a whole number) are modeled by differential equations which are intrinsically continuous. To solve the resulting differential equations, they are turned into discrete difference equations to be solved by a computer. Since the original problem is discrete, it would seem more reasonable to use difference equations from the start! And computers make this approach quite feasible. Difference equations are rapidly gaining in importance.