4.3 Inverse Functions and their Derivatives

In this unit you’ll review inverse functions, how to find them, and how the graphs of functions and their inverses are related geometrically. Not all functions can be “undone”, but when they can be, the inverse function undoes whatever the function did. If the function ties a knot, then the inverse function untyes the knot. Then we’ll apply the chain rule to find the derivatives of inverse functions.

Inverse Functions

First let \( f \) be the function
\[
f = \{ (1,2), (2,4), (3,6), (4,1), (5,3), (6,5) \}
\]
with arrow diagram:

Since the inverse function \( f^{-1} \) undoes what \( f \) did, it’s easy to find the arrow diagram for this inverse function; just draw in the arrows that clean up the mess \( f \) created, the arrow diagram that “undoes” the function \( f \):

This diagram was constructed by thinking:
\[
f \text{ sends } 1 \text{ to } 2, \text{ so } f^{-1} \text{ sends 2 back to 1.}
\]
\[
f \text{ sends } 2 \text{ to } 4, \text{ so } f^{-1} \text{ sends 4 back to 2.}
\]
Etc…

Thus,
\[
f^{-1} = \{ (2,1), (4,2), (6,3), (1,4), (3,5), (5,6) \}.
\]

Notice that the points in \( f^{-1} \) are simply the points in \( f \), but with all their \( xy \)-coordinates switched! For example, in place of the point (1,2), the inverse function has the point (2,1). Below is a graph of function \( f \), along with its inverse function, \( f^{-1} \).

Plotting \( f \) in blue, and its inverse \( f^{-1} \) in red, we get the diagram to the right:

The diagonal line \( y=x \) is graphed in yellow; notice the symmetry of the graphed points about that line.
Important concept!

Switching the \(xy\)-coordinates of any point reflects that point through the diagonal line \(y=x\), as illustrated in the diagram:

Since all the points of the graph of the inverse function are points of the original function with the \(xy\)-coordinates switched, it follows that the graph of the inverse function is the graph of the function reflected through the line \(y=x\), as we saw to be the case in the above example.

Our simple example of a function and its inverse worked because not only did our original function pass the “vertical line test” (vertical lines cross the graph of ANY function in at most one place), but it also passes the “horizontal line test” (horizontal lines cross the graph of the function in at most one place). In such cases the function is said to be “one-to-one”, which means each input to the function (those are the \(x\)-coordinates in the points that make up the graph of the function) is paired with a unique output from the function (those are the \(y\)-coordinates in the points that make up the graph of the function), and also each output from the function has a unique input that generates it.

IMPORTANT! The inverse of a function is not its reciprocal, in general: \(f^{-1} \neq \frac{1}{f}\).

The notation is unfortunately ambiguous, but whenever you see \(f^{-1}(x)\), this means the inverse function and not the reciprocal of the function.

Most functions we need deal with are defined in terms of equations, usually where \(y\) is expressed algebraically as a function of \(x\). Finding the inverse function algebraically also involves switching the \(x\) and \(y\) coordinates. For example, to find the inverse function of the function \(f(x) = 3x - 5\), we start by switching the \(x\)’s and \(y\)’s:

\[
y = 3x - 5 \quad \Rightarrow \quad x = 3y - 5
\]

Then solve the resulting “switched” equation for \(y\):

\[
x + 5 = 3y \quad \Rightarrow \quad \frac{x + 5}{3} = y.
\]

That’s the inverse function, \(f^{-1}(x) = \frac{x + 5}{3}\).
Here are both functions plotted on the same graph, along with the line $y=x$.

$$f^{-1}(x) = \frac{x + 5}{3}$$

$y = x$

$f(x) = 3x - 5$

NOTE: To find an inverse function when the equation for the function expresses $y$ as a function of $x$, we’ve first switched the $x$’s and $y$’s in the equation, and then solved for $y$. But you can also do this in the reverse order, first solving for $x$ as a function of $y$ and then switching the $x$’s and $y$’s in the equation.

Examples:

1. Let $y = \frac{2}{3x + 1}$. Find $\frac{dy}{dx}$ in terms of $x$ and $\frac{dx}{dy}$ in terms of $y$.

We are given $y$ as a function of $x$: $y = f(x) = \frac{2}{3x + 1}$.

$$y = \frac{2}{3x + 1} = 2(3x + 1)^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx} 2(3x + 1)^{-1} = -2(3x + 1)^{-2} \cdot \frac{d}{dx}(3x + 1) = -2(3x + 1)^{-2} \cdot 3 = \frac{-6}{(3x + 1)^{2}} = f'(x)$$

Solving for $x$ as a function of $y$, we get:

$$y = \frac{2}{3x + 1} \quad \Rightarrow \quad y(3x + 1) = 2 \quad \Rightarrow \quad 3xy + y = 2 \quad \Rightarrow \quad 3xy = 2 - y \quad \Rightarrow \quad x = \frac{2 - y}{3y}$$

That last line is the derivation of the inverse function, $x = f^{-1}(y) = \frac{2 - y}{3y}$.

Differentiating with respect to $y$, we get:

$$\frac{d}{dy} x = \frac{d}{dy} \frac{2 - y}{3y} = \frac{(3y) \cdot \frac{d}{dy} (2 - y) - (2 - y) \cdot \frac{d}{dy} (3y)}{(3y)^2} = \frac{3y \cdot (-1) - (2 - y) \cdot 3}{(3y)^2} = \frac{-3y - 6 + 3y}{(3y)^2} = \frac{-6}{9y^2} = \frac{-2}{3y^2} = \frac{dx}{dy} = f^{-1}'(y)$$
Notice that the equation \( f^{-1}(y) = \frac{2-y}{3y} \) defines the function \( f^{-1} \) just as well as the equation \( f^{-1}(x) = \frac{2-x}{3x} \). It’s the same function either way! 😊

Plotting
\[
f(x) = \frac{2}{3x+1} \text{ in blue,}
\]
and its inverse
\[
f^{-1}(x) = \frac{2-x}{3x} \text{ in red,}
\]
we get the diagram to the right:
The diagonal line \( y = x \) is graphed in yellow; notice the symmetry of the graphed functions about that line.

2. Let \( y = f(x) = \frac{ax + b}{cx + d} \), and let \( x = f^{-1}(y) \). Find \( \frac{dy}{dx} \) in terms of \( x \) and \( \frac{dx}{dy} \) in terms of \( y \).

\[
y = \frac{ax + b}{cx + d} \Rightarrow \frac{dy}{dx} = \frac{d}{dx} \frac{ax + b}{cx + d} = \frac{(cx+d) \cdot \frac{d}{dx} (ax + b) - (ax + b) \cdot \frac{d}{dx} (cx + d)}{(cx + d)^2}
\]

\[
= \frac{(cx+d) \cdot a - (ax+b) \cdot c}{(cx + d)^2} = \frac{acx + ad - acx - bc}{(cx + d)^2} = \frac{ad - bc}{(cx + d)^2}
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{ad - bc}{(cx + d)^2}
\]

Solving for \( x \) as a function of \( y \) (and thus finding the inverse function), we get:

\[
y = \frac{ax + b}{cx + d} \Rightarrow y(cx + d) = ax + b \Rightarrow cxy + dy = ax + b \Rightarrow cxy - ax = b - dy
\]

\[
\Rightarrow (cy - a)x = b - dy \Rightarrow x = \frac{b - dy}{cy - a} = f^{-1}(y)
\]

Differentiating with respect to \( y \), we get:

\[
\frac{d}{dy} x = \frac{d}{dy} \frac{b - dy}{cy - a} = \frac{(cy-a) \cdot \frac{d}{dy} (b-dy) - (b-dy) \cdot \frac{d}{dy} (cy-a)}{(cy-a)^2} = \frac{(cy-a) \cdot (-d) - (b-dy) \cdot (-d)}{(cy-a)^2}
\]

\[
= \frac{cdy + ad - bc + cdy}{(cy-a)^2} \Rightarrow \frac{dx}{dy} = \frac{ad - bc}{(cy-a)^2} = \frac{d}{dy} f^{-1}(y)
\]
The Derivative of an Inverse Function

To find the derivative of an inverse function, \( \frac{d}{dx} \left[ f^{-1}(x) \right] \), we begin with the algebraic equation that describes the fact that the function \( f \) undoes whatever the function \( f^{-1} \) does to \( x \) (since these functions are inverses of each other), and then we differentiate both sides of that equation:

\[
f \left( f^{-1}(x) \right) = x
\]

\[
\frac{d}{dx} f \left( f^{-1}(x) \right) = \frac{d}{dx} x
\]

\[
\frac{d}{dx} f \left( f^{-1}(x) \right) = 1
\]

Now we must invoke the chain rule, \( \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \), to get

\[
f' \left( f^{-1}(x) \right) \cdot \frac{d}{dx} f^{-1}(x) = 1
\]

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f' \left( f^{-1}(x) \right)}
\]

If we let \( y = f(x) \) and \( x = f^{-1}(y) \), then the above equation can be expressed more concisely:

\[
\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}
\]

In the next section we’ll use this technique to derive formulas for the inverse trigonometric functions.

Derivative Examples:

3. Let \( y = \tan x \). Find \( \frac{dx}{dy} \) at \( x = \frac{\pi}{4} \).

At \( x = \frac{\pi}{4} \), we have \( y = \tan \frac{\pi}{4} = 1 \), so we want to find the derivative of the inverse function of \( y = \tan x \), at the point \( y = 1 \). This is the reciprocal of the derivative \( \frac{dy}{dx} \) evaluated at \( x = \frac{\pi}{4} \).

But \( \frac{dy}{dx} \bigg|_{x = \frac{\pi}{4}} = \frac{d}{dx} \tan x \bigg|_{x = \frac{\pi}{4}} = (\sec x)^2 \bigg|_{x = \frac{\pi}{4}} = \left( \sec \frac{\pi}{4} \right)^2 = (\sqrt{2})^2 = 2 \).
And so, \[
\frac{dx}{dy} \bigg|_{y=1} = \frac{1}{\frac{dy}{dx}} \bigg|_{x=\frac{\pi}{4}} = \frac{1}{2}
\]

4. Let \( y = x - \sin x \). Find \( \frac{dx}{dy} \) at \( x = 0 \).

At \( x = 0 \), we have \( y = 0 - \sin 0 = 0 \), so we want to find the derivative of the inverse function of \( y = x - \sin x \), at the point \( y = 0 \). This is the reciprocal of the derivative \( \frac{dy}{dx} \) evaluated at \( x = 0 \).

But \[
\frac{dy}{dx} \bigg|_{x=0} = \frac{d}{dx} (x - \sin x) \bigg|_{x=0} = (1 - \cos x) \bigg|_{x=0} = 1 - \cos 0 = 1 - 1 = 0.
\]

The inverse function’s derivative is then the reciprocal of 0, which is undefined. At any points on the graph of \( y = f(x) \) with horizontal tangent lines, the corresponding points on the graph of \( y = f^{-1}(x) \) have vertical tangent lines. The derivative of the inverse function isn’t defined at such points.

Plotting \( y = f(x) = x - \sin x \) in blue, and its inverse \( y = f^{-1}(x) \) in red, we get the diagram to the right:

The diagonal line \( y = x \) is graphed in yellow; notice the symmetry of the graphed functions about that line. Also notice that at \((0,0)\), \( f \) has a horizontal tangent (the \( x \)-axis) while \( f^{-1} \) has a vertical tangent (the \( y \)-axis).