4.4 Derivatives of the Inverse Trigonometric Functions

In the previous section, we went over the basic process of finding the derivative of an inverse function:

\[
f^{-1}(x) = x
\]

\[
\frac{d}{dx} f^{-1}(x) = 1
\]

\[
f^{-1}(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1
\]

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}
\]

In this section, we’ll go through this process for each of the trigonometric functions. Recall that the trigonometric functions are all periodic, so are not one-to-one. But we can restrict their domains so that over the restricted domain they are one-to-one. Unless otherwise stated, all “angles” are in radians, which are simply real numbers.

The Arc Sine

The inverse of the sine function (with its domain restricted to the interval \([-\frac{\pi}{2}, \frac{\pi}{2}\]) is the inverse sine function, \(\sin^{-1}\), also known as the arc sine. Since the sine over that restricted domain has range \([-1, 1]\), it follows that the domain of the arc sine is \([-1, 1]\) with range \([-\frac{\pi}{2}, \frac{\pi}{2}\]). It’s important to think of \(\sin^{-1}(a)\) as the angle in \([-\frac{\pi}{2}, \frac{\pi}{2}]\) that has sine \(a\).

Plot of \(y = \sin x\) in blue, and its inverse \(y = \sin^{-1} x = \arcsin x\) in red:

The diagonal line \(y=x\) is graphed in yellow; notice the symmetry of the graphed functions about that line.

Also notice that the domain of each of these functions is the range of the other, and vice versa.

To find the derivative of the arc sine function, follow the above steps with \(f(x) = \sin(x)\) and \(f^{-1}(x) = \sin^{-1}(x)\):
\[
\sin\left(\sin^{-1}(x)\right) = x
\]
\[
\frac{d}{dx} \sin\left(\sin^{-1}(x)\right) = 1
\]
\[
\cos\left(\sin^{-1}(x)\right) \frac{d}{dx} \sin^{-1}(x) = 1
\]
\[
\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos\left(\sin^{-1}(x)\right)}
\]

But we can go a little further, if we recall some trigonometry. Now \(\sin^{-1}(x)\) is that angle that has sine \(x\). Let’s give this angle a simpler name; let \(\theta = \sin^{-1}(x)\). Then we want to know what \(\cos \theta\) equals, given that \(\sin \theta = x\). Draw a simple right triangle with an angle \(\theta\) that has \(\sin \theta = x\):

\[
\text{This triangle is the right one:}
\]
\[
\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{x}{1} = x
\]

Fill in the missing leg:

\[
\text{Now we see that} \quad \cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.
\]
Taking our arc sine derivative one step further, we obtain:
\[
\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos\left(\sin^{-1}(x)\right)} = \frac{1}{\cos \theta} = \frac{1}{\sqrt{1-x^2}}
\]
\[
\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}
\]

Here \(x\) must be restricted to the interval \((-1,1)\), since the derivative of the arc sine isn’t defined at the endpoints of this interval, where the arc sine’s tangent lines become vertical.

As always, a derivative formula yields a corresponding integral formula, via the fundamental theorem of calculus. Thus we have:
\[
\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C, \quad \text{for} \quad x \in (-1,1).
\]

**The Arc Cosine**

The inverse of the cosine function (with its domain restricted to the interval \([0, \pi]\)) is the **inverse cosine function**, \(\cos^{-1}\), also known as the **arc cosine**. Since the cosine over that restricted domain has range \([-1,1]\), it follows that the domain of the arc cosine is \([-1,1]\) with range \([0,\pi]\). It’s important to think of \(\cos^{-1}(a)\) as the angle in \([0,\pi]\) that has cosine \(a\).
Plot of \( y = \cos x \) in blue, and its inverse \( y = \cos^{-1} x = \arccos x \) in red:

The diagonal line \( y = x \) is graphed in yellow; notice the symmetry of the graphed functions about that line.

Also notice that the domain of each of these functions is the range of the other, and vice versa.

To find the derivative of the arc cosine function, follow the above steps with \( f(x) = \cos x \) and \( f^{-1}(x) = \cos^{-1}(x) \):

\[
\cos\left(\cos^{-1}(x)\right) = x
\]
\[
\frac{d}{dx} \cos\left(\cos^{-1}(x)\right) = 1
\]
\[
-\sin\left(\cos^{-1}(x)\right) \frac{d}{dx} \cos^{-1}(x) = 1
\]
\[
\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sin\left(\cos^{-1}(x)\right)}
\]

But we can go a little further, if we recall some trigonometry. Now \( \cos^{-1}(x) \) is that angle that has cosine \( x \). Let’s give this angle a simpler name; let \( \theta = \cos^{-1}(x) \). Then we want to know what \( \sin \theta \) equals, given that \( \cos \theta = x \). Draw a simple right triangle with an angle \( \theta \) such that \( \cos \theta = x \):

This triangle is the right one:

\[
\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x
\]

Fill in the missing leg:

Now we see that \( \sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \). Taking our arc cosine derivative one step further, we obtain:

\[
\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sin\left(\cos^{-1}(x)\right)} = \frac{-1}{\sin \theta} = \frac{-1}{\sqrt{1-x^2}}
\]
\[
\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}
\]
Here \( x \) must be restricted to the interval \((-1,1)\), since the derivative of the arc cosine isn’t defined at the endpoints of this interval, where the arc cosine’s tangent lines become vertical.

As always, a derivative formula yields a corresponding integral formula, via the fundamental theorem of calculus. Thus we have:

\[
\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C, \text{ for } x \in (-1,1).
\]

Since this is basically the same as the previous integral formula, you only need remember one of them; the arc sine formula is slightly nicer, since it lacks the negative sign.

**The Arc Tangent**

The inverse of the tangent function, with its domain restricted to the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\), is the inverse tangent function, \( \tan^{-1} \), also known as the arc tangent. Since the tangent over that restricted domain has range all real numbers \((-\infty, \infty)\), it follows that the domain of the arc tangent is \((-\infty, \infty)\) with range \((-\frac{\pi}{2}, \frac{\pi}{2})\). It’s important to think of \( \tan^{-1}(a) \) as the angle in \((-\frac{\pi}{2}, \frac{\pi}{2})\) that has tangent \(a\).

Plot of \( y = \tan x \) in blue, and its inverse \( y = \tan^{-1} x = \arctan(x) \) in red:

The diagonal line \( y = x \) is graphed in yellow; notice the symmetry of the graphed functions about that line.

Also notice that the domain of each of these functions is the range of the other, and vice versa.

And also notice those green asymptotic lines!

To find the derivative of the arc tangent function, follow the above steps with \( f(x) = \tan x \) and \( f^{-1}(x) = \tan^{-1}(x) \):

\[
\tan(\tan^{-1}(x)) = x
\]

\[
\frac{d}{dx} \tan(\tan^{-1}(x)) = 1
\]

\[
\sec^2(\tan^{-1}(x)) \frac{d}{dx} \tan^{-1}(x) = 1
\]

\[
\frac{d}{dx} \tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))}
\]
But we can go a little further, recalling a little trigonometry. Now \( \tan^{-1}(x) \) is that angle that has a tangent of \( x \). Let’s give this angle a simpler name; let \( \theta = \tan^{-1}(x) \). Then we want to know what \( \sec \theta \) equals, given that \( \tan \theta = x \). Draw a simple right triangle with an angle \( \theta \) that has \( \tan \theta = x \):

![Diagram of a right triangle](image)

This triangle is the right one:

\[
\tan \theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{x}{1} = x
\]

Fill in the missing leg:

Now we see that \( \sec \theta = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{\sqrt{x^2+1}}{1} = \sqrt{x^2+1} \). Taking our arc tangent derivative one step further, we get:

\[
\frac{d}{dx} \tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{\left(\sec(\tan^{-1}(x))\right)^2} = \frac{1}{\left(\sqrt{x^2+1}\right)^2} = \frac{1}{x^2+1}
\]

Here \( x \) can be any real number, \((-\infty, \infty)\), since the domain of the arc tangent is all real numbers.

As always, a derivative formula yields a corresponding integral formula, via the fundamental theorem of calculus. Thus we have:

\[
\int \frac{dx}{x^2+1} = \tan^{-1}(x) + C, \text{ for } x \in (-\infty, \infty).
\]

**The Arc Cotangent**

The inverse of the cotangent function, with its domain restricted to the interval \((0, \pi)\), is the *inverse cotangent function*, \(\cot^{-1}\), also known as the *arc cotangent*. Since the cotangent over that restricted domain has range all real numbers \((-\infty, \infty)\), it follows that the domain of the arc cotangent is \((-\infty, \infty)\) with range \((0, \pi)\). It’s important to think of \(\cot^{-1}(a)\) as the angle in \((0, \pi)\) that has cotangent \(a\).

Plot of \( y = \cot x \) in blue, and its inverse, \( y = \cot^{-1} x = \arccot(x) \) in red:

The diagonal line \( y=x \) is graphed in yellow; notice the symmetry of the graphed functions about that line.

Also notice that the domain of each of these functions is the range of the other, and vice versa.
To find the derivative of the arc cotangent function, follow the above steps with \( f(x) = \cot x \) and \( f^{-1}(x) = \cot^{-1}(x) \):

\[
\cot(\cot^{-1}(x)) = x
\]

\[
\frac{d}{dx}\cot(\cot^{-1}(x)) = 1
\]

\[
-\csc^2(\cot^{-1}(x))\frac{d}{dx}\cot^{-1}(x) = 1
\]

\[
\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{\csc^2(\cot^{-1}(x))}
\]

But we can go further, recalling some trig. Now \( \cot^{-1}(x) \) is that angle that has a cotangent of \( x \). Let’s give this angle a simpler name; let \( \theta = \cot^{-1}(x) \). Then we want to know what \( \csc \theta \) equals, given that \( \cot \theta = x \). Draw a simple right triangle with an angle \( \theta \) such that \( \cot \theta = x \):

This triangle is the right one:

\[
\cot \theta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{x}{1} = x
\]

Fill in the missing leg:

\[
\text{hypotenuse} = \sqrt{x^2 + 1}
\]

Now we see that \( \csc \theta = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{\sqrt{x^2 + 1}}{1} = \sqrt{x^2 + 1} \). Taking our arc cotangent derivative one step further, we obtain:

\[
\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{\csc^2(\cot^{-1}(x))} = \frac{-1}{(\csc(\cot^{-1}(x)))^2} = \frac{-1}{(\sqrt{x^2 + 1})^2} = \frac{-1}{x^2 + 1}
\]

\[
\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{x^2 + 1}
\]

Here \( x \) can be any real number, \((-\infty, \infty)\), since the domain of the arc cotangent is all real numbers.

As always, a derivative formula yields a corresponding integral formula, via the fundamental theorem of calculus. Thus we have:

\[
\int \frac{dx}{x^2 + 1} = -\cot^{-1}(x) + C, \quad \text{for } x \in (-\infty, \infty).
\]

This is basically the same as the previous integral formula involving the \( \tan^{-1}(x) \), so you only need remember one of them. The previous arc tangent formula is slightly nicer, since it lacks the negative sign.
The Arc Secant

The inverse of the secant function, with its domain restricted to the union of intervals \( \left[ 0, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \pi \right] \), is the inverse secant function, \( \sec^{-1} \), also known as the arc secant. Since the secant over that restricted domain has range \( (-\infty, -1] \cup [1, \infty) \), it follows that the domain of the arc secant is \( (-\infty, -1] \cup [1, \infty) \) with range \( \left[ 0, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \pi \right] \). It’s important to think of \( \sec^{-1}(a) \) as the angle in \( \left[ 0, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \pi \right] \) that has secant \( a \). The domain of the arc secant can be more simply expressed as \( \left\{ x \mid |x| \geq 1 \right\} \).

Plot of \( y = \sec x \) in blue, and its inverse \( y = \sec^{-1} x = \text{arcsec}(x) \) in red:

The diagonal line \( y=x \) is graphed in yellow; notice the symmetry of the graphed functions about that line. The green lines are asymptotes.

Also notice that the domain of each of these functions is the range of the other, and vice versa.

To find the derivative of the arc secant function, follow the above steps with \( f(x) = \sec x \) and \( f^{-1}(x) = \sec^{-1}(x) \) (recalling that the derivative of the secant is the secant times the tangent):

\[
\sec(\sec^{-1}(x)) = x \\
\frac{d}{dx}\sec(\sec^{-1}(x)) = 1 \\
\sec(\sec^{-1}(x)) \cdot \tan(\sec^{-1}(x)) \cdot \frac{d}{dx}\sec^{-1}(x) = 1 \\
x \cdot \tan(\sec^{-1}(x)) \cdot \frac{d}{dx}\sec^{-1}(x) = 1 \\
\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x \tan(\sec^{-1}(x))}
\]

But we can go further. Now \( \sec^{-1}(x) \) is that angle that has secant \( x \). Let’s give this angle a simpler name; let \( \theta = \sec^{-1}(x) \). Then we want to know what \( \tan \theta \) equals, given that \( \sec \theta = x \). Draw a simple right triangle with an angle \( \theta \) such that \( \sec \theta = x \):

This triangle is the right one:

\[
\text{sec} \theta = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{x}{1} = x
\]

Fill in the missing leg:
Now we see that \( \tan \theta = \frac{\text{side opposite}}{\text{side adjacent}} = \pm \sqrt{x^2 - 1} \) and so
\[
\frac{d}{dx} \sec^{-1}(x) = \frac{1}{x \tan(\sec^{-1}(x))} = \pm \frac{1}{x\sqrt{x^2 - 1}}.
\]

Notice from the graph of the arc secant that its slope is always positive. Thus we pick the + when \( x \) is positive, and the – when \( x \) is negative, to make sure the derivative remains positive. Or more simply,
\[
\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2 - 1}}
\]

Here \( x \) must be restricted to values for which \( |x| > 1 \), since the derivative of the arc secant isn’t defined at \( x = \pm 1 \), where the arc secant’s tangent lines become vertical.

As always, a derivative formula yields a corresponding integral formula, via the fundamental theorem of calculus. Thus we have:
\[
\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C, \quad \text{for} \ |x| > 1.
\]

**The Arc Cosecant**

The inverse of the cosecant function, with its domain restricted to the union of intervals \([-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\), is the *inverse cosecant function*, \( \csc^{-1} \), also known as the *arc cosecant*. Since the cosecant over that restricted domain has range \((-\infty, -1] \cup [1, \infty)\), it follows that the domain of the arc cosecant is \((-\infty, -1] \cup [1, \infty)\) with range \(\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]\). It’s important to think of \( \csc^{-1}(a) \) as the angle in \(\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]\) that has cosecant \(a\). The domain of the arc cosecant also = \(\left\{x \mid |x| \geq 1\right\}\).

Plot of \( y = \csc x \) in blue, and its inverse \( y = \csc^{-1} x = \arccsc(x) \) in red:

The diagonal line \( y = x \) is graphed in yellow; notice the symmetry of the graphed functions about that line. The green lines are asymptotes.

Also notice that the domain of each of these functions is the range of the other, and vice versa.
To find the derivative of the arc cosecant function, follow the above steps with

\[ f(x) = \csc x \quad \text{and} \quad f^{-1}(x) = \csc^{-1}(x). \]

(recalling that the derivative of the cosecant is the negative of the secant times the tangent):

\[
\csc \left( \csc^{-1}(x) \right) = x \\
\frac{d}{dx} \csc \left( \csc^{-1}(x) \right) = 1 \\
- \csc \left( \csc^{-1}(x) \right) \cdot \cot \left( \csc^{-1}(x) \right) \cdot \frac{d}{dx} \csc^{-1}(x) = 1 \\
- x \cdot \cot \left( \csc^{-1}(x) \right) \cdot \frac{d}{dx} \csc^{-1}(x) = 1 \\
\frac{d}{dx} \csc^{-1}(x) = \frac{-1}{x \cot \left( \csc^{-1}(x) \right)}
\]

But we can go further. Now \( \csc^{-1}(x) \) is that angle that has secant \( x \). Let’s give this angle a simpler name; let \( \theta = \csc^{-1}(x) \). Then we want to know what \( \cot \theta \) equals, given that \( \csc \theta = x \). Draw a simple right triangle with an angle \( \theta \) such that \( \csc \theta = x \):

This triangle is the right one:

\[
\csc \theta = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{x}{1} = x \\
\text{Fill in the missing leg:}
\]

Now we see that \( \cot \theta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{\pm \sqrt{x^2 - 1}}{1} = \pm \sqrt{x^2 - 1} \). Taking our arc cosecant derivative one step further, we obtain:

\[
\frac{d}{dx} \csc^{-1}(x) = \frac{-1}{x \cot \left( \csc^{-1}(x) \right)} = \frac{\pm 1}{x \sqrt{x^2 - 1}}
\]

Notice from the graph of the arc cosecant that its slope is always negative. Thus we pick the “–” when \( x \) is positive, and the “+” when \( x \) is negative, to make sure the derivative remains positive. Or more simply,

\[
\frac{d}{dx} \csc^{-1} x = \frac{-1}{x |x| \sqrt{x^2 - 1}}
\]

Here \( x \) must be restricted to values for which \( |x| > 1 \), since the derivative of the arc cosecant isn’t defined at \( x = \pm 1 \), where the arc cosecant’s tangent lines become vertical.

As always, a derivative formula yields a corresponding integral formula, via the fundamental theorem of calculus. Thus we have:

\[
\int \frac{1}{|x| \sqrt{x^2 - 1}} = -\csc^{-1} x + C, \quad \text{for} \ |x| > 1.
\]
This is basically the same as the previous integral formula involving the $\sec^{-1}(x)$, so you only need remember one of them. The previous arc secant formula is slightly nicer, since it lacks the negative sign.

To summarize, we now have six new differentiation formulas and three new integral formulas:

<table>
<thead>
<tr>
<th>Differentiation Formula</th>
<th>Integral Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, for $x \in (-1,1)$</td>
<td>$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$, for $x \in (-1,1)$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$, for $x \in (-1,1)$</td>
<td></td>
</tr>
<tr>
<td>$\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$, for $x \in (-\infty, \infty)$</td>
<td>$\int \frac{dx}{x^2+1} = \tan^{-1} x + C$, for $x \in (-\infty, \infty)$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \sec^{-1} x = \frac{1}{</td>
<td>x</td>
</tr>
<tr>
<td>$\frac{d}{dx} \csc^{-1} x = \frac{-1}{</td>
<td>x</td>
</tr>
</tbody>
</table>

Generalized derivative rules:

Start with our new derivative formula, $\frac{d}{dx} \sin^{-1} (x) = \frac{1}{\sqrt{1-x^2}}$, and generalize it by apply the chain rule:

$$\frac{d}{dx} \sin^{-1} (f(x)) = \frac{1}{\sqrt{1-(f(x))^2}} \cdot f'(x) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}.$$  If we let $f(x) = u$, then this formula can be expressed more simply as

$$\frac{d}{dx} \sin^{-1} u = \frac{u'}{\sqrt{1-u^2}}.$$  Similarly one may generalize each of the previous derivative formulas – simply replace $x$ by $u$ in a formula, and then multiply times $u'$. 
Here are the “generalized” inverse trigonometric function derivative formulas:

| Derivative of $\sin^{-1} u$, $\cos^{-1} u$, $\tan^{-1} u$, $\cot^{-1} u$, $\sec^{-1} u$, $\csc^{-1} u$ |
|-----------------|-----------------|
| $\frac{d}{dx} \sin^{-1} u = \frac{u'}{\sqrt{1-u^2}}$, for $u \in (-1,1)$ | $\frac{d}{dx} \cos^{-1} u = \frac{-u'}{\sqrt{1-u^2}}$, for $u \in (-1,1)$ |
| $\frac{d}{dx} \tan^{-1} u = \frac{u'}{u^2+1}$, for $u \in (-\infty, \infty)$ | $\frac{d}{dx} \cot^{-1} u = \frac{-u'}{u^2+1}$, for $u \in (-\infty, \infty)$ |
| $\frac{d}{dx} \sec^{-1} u = \frac{u'}{|u|\sqrt{u^2-1}}$, for $|u|>1$ | $\frac{d}{dx} \csc^{-1} u = \frac{-u'}{|u|\sqrt{u^2-1}}$, for $|u|>1$ |

**Derivative Examples**

1. If $y = \sin^{-1}(2x^3)$, find $\frac{dy}{dx}$.

   \[
   \frac{dy}{dx} = \frac{d}{dx} \sin^{-1}(2x^3) = \frac{\frac{d}{dx} 2x^3}{\sqrt{1-(2x^3)^2}} = \frac{6x^2}{\sqrt{1-4x^6}}
   \]

2. If $y = \cos^{-1}(\tan x)$, find $\frac{dy}{dx}$.

   \[
   \frac{dy}{dx} = \frac{d}{dx} \cos^{-1}(\tan x) = \frac{-\frac{d}{dx} \tan x}{\sqrt{1-(\tan x)^2}} = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}
   \]

3. If $f(x) = \sin^{-1}(x^2 + \cos x)$, then $f' \left( \frac{\pi}{2} \right)$ is?

   \[
   f'(x) = \frac{d}{dx} \sin^{-1}(x^2 + \cos x) = \frac{\frac{d}{dx} (x^2 + \cos x)}{\sqrt{1-(x^2 + \cos x)^2}} = \frac{2x - \sin x}{\sqrt{1-(x^2 + \cos x)^2}}
   \]

   \[
   f' \left( \frac{\pi}{2} \right) = \frac{2 \cdot \frac{\pi}{2} - \sin \frac{\pi}{2}}{\sqrt{1-\left( \left( \frac{\pi}{2} \right)^2 + \cos \frac{\pi}{2} \right)^2}} = \frac{\pi - 1}{\sqrt{1-\left( \left( \frac{\pi}{2} \right)^2 + 0 \right)^2}} = \frac{\pi - 1}{\sqrt{1-\left( \frac{\pi^2}{4} \right)^2}} = \frac{\pi - 1}{\sqrt{1-\frac{\pi^4}{16}}} = \frac{4(\pi - 1)}{\sqrt{16-\pi^4}}
   \]
4. If \( g(x) = \cos^{-1}\left(\sin^2 x\right) \), then \( g'\left(\frac{\pi}{4}\right) = ? \)

\[
g'(x) = \frac{d}{dx} \cos^{-1}\left(\sin^2 x\right) = \frac{-\frac{d}{dx} \sin^2 x}{\sqrt{1-(\sin^2 x)^2}} = \frac{-2 \sin x \cos x}{\sqrt{1-\sin^4 x}}
\]

\[
g'\left(\frac{\pi}{4}\right) = \frac{-2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}}{\sqrt{1-\sin^4 \left(\frac{\pi}{4}\right)}} = \frac{-2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} = \frac{-1}{\sqrt{\frac{1}{4}}} \cdot \frac{3}{\sqrt{4}} = -2
\]

5. If \( y = \tan^{-1}(x^3 + x^2 + 1) \), find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{d}{dx} \tan^{-1}(x^3 + x^2 + 1) = \frac{\frac{d}{dx}(x^3 + x^2 + 1)}{(x^3 + x^2 + 1)^2 + 1} = \frac{3x^2 + 2x}{(x^3 + x^2 + 1)^2 + 1}
\]

6. If \( h(x) = \tan^{-1}\left(\tan^{-1} x\right) \), then \( h'(1) = ? \)

\[
h''(x) = \frac{d}{dx} \tan^{-1}\left(\tan^{-1} x\right) = \frac{\frac{d}{dx}\left(\tan^{-1} x\right)}{(\tan^{-1} x)^2 + 1} = \frac{1}{x^2 + 1} = \frac{1}{(x^2 + 1)((\tan^{-1} x)^2 + 1)}
\]

\[
h'(1) = \frac{1}{(1^2 + 1)((\tan^{-1} 1)^2 + 1)} = \frac{1}{2\left(\left(\frac{\pi}{4}\right)^2 + 1\right)} = \frac{1}{2\left(\frac{\pi^2}{16} + 1\right)} = \frac{8}{\pi^2 + 16}
\]

7. If \( y = \cot^{-1}(x + \sin x) \), find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{d}{dx} \cot^{-1}(x + \sin x) = \frac{-\frac{d}{dx}(x + \sin x)}{(x + \sin x)^2 + 1} = \frac{-(1 + \cos x)}{(x + \sin x)^2 + 1}
\]

8. If \( f(x) = \cot^{-1}\left(\tan^{-1}(2x)\right) \), then \( f'\left(\frac{1}{2}\right) = ? \)

\[
f'(x) = \frac{d}{dx} \cot^{-1}\left(\tan^{-1}(2x)\right) = \frac{-\frac{d}{dx}\left(\tan^{-1}(2x)\right)}{(\tan^{-1}(2x))^2 + 1} = \frac{-\frac{d}{dx}(2x)}{(2x)^2 + 1} = \frac{-\frac{d}{dx}(2x)}{(2x)^2 + 1}
\]
\[
f'(1) = \frac{-2}{(4 \left(\frac{1}{2}\right)^2 + 1) \left(\tan^{-1}\left(2 \cdot \frac{1}{2}\right)\right)^2 + 1} = -2 = -16 \\
\pi^2 + 16
\]

9. If \( y = \sec^{-1}\left(\sqrt{x^2 + 1}\right) \), find \( \frac{dy}{dx} \). (Assume that \( x > 0 \))

\[
y' = \frac{d}{dx} \sec^{-1}\left(\sqrt{x^2 + 1}\right) = \frac{d}{dx} \left(x^2 + 1\right)^{\frac{1}{2}} = \frac{1}{2} \left(x^2 + 1\right)^{\frac{1}{2}} \cdot 2x
\]

\[
y' = \frac{\left(x^2 + 1\right)^{\frac{1}{2}} \cdot x}{\sqrt{x^2 + 1} \cdot x} = \frac{1}{x^2 + 1}
\]

Notice that \( \frac{d}{dx} \tan^{-1}x = \frac{d}{dx} \sec^{-1}\left(\sqrt{x^2 + 1}\right) \). When the derivatives of two functions are equal, the functions are equal up to a constant. In other words, \( \sec^{-1}\left(\sqrt{x^2 + 1}\right) = \tan^{-1}x + C \) for some constant \( C \). Indeed, that constant is 0, so for \( x > 0 \), \( \sec^{-1}\left(\sqrt{x^2 + 1}\right) = \tan^{-1}x \). That surprised me!!

10. If \( Q(x) = \csc^{-1}\left(\pi \tan^{-1}x\right) \), then \( Q'(1) = ? \). (Assume that \( x > 0 \))

\[
Q'(x) = \frac{d}{dx} \csc^{-1}\left(\pi \tan^{-1}x\right) = \frac{- \frac{d}{dx} \pi \tan^{-1}x}{\pi \tan^{-1}x \sqrt{\left(\pi \tan^{-1}x\right)^2 - 1}} = \frac{- \frac{1}{x^2 + 1}}{\sqrt{\left(\pi \tan^{-1}x\right)^2 - 1}}
\]

\[
Q'(1) = \frac{-1}{\left(\tan^{-1}1\right)(1^2 + 1) \sqrt{\left(\pi \tan^{-1}1\right)^2 - 1}} = \frac{-1}{\left(\pi \left(\frac{1}{2}\right)\right)^2 \sqrt{\left(\pi \cdot \frac{\pi}{4}\right)^2 - 1}} = \frac{-1}{\pi \sqrt{\pi^4 - 16}} = -8
\]
11. If $y = \sin^{-1} x + \cos^{-1} x$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

Notice that $\frac{dy}{dx} = 0$ for all $x$. This implies that $y = \sin^{-1} x + \cos^{-1} x = C$ for some constant $C$.

Plugging in $x=0$, we can find that constant: $C = \sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$. Thus, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

For all $x$ in the domain of both of these inverse trig functions. Calculus provides new methods for proving trigonometric identities!

12. If $y = \sqrt{\tan^{-1}(3x)}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(3x))^{\frac{1}{2}} = \frac{1}{3}(\tan^{-1}(3x))^{\frac{1}{2}} \cdot \frac{d}{dx}(\tan^{-1}(3x)) = \frac{1}{3\sqrt{(\tan^{-1}(3x))^2}} \cdot \frac{d}{dx}(3x)$$

$$= \frac{1}{3\sqrt{(\tan^{-1}(3x))^2}} \cdot \frac{3}{9x^2+1} = \frac{1}{(9x^2+1)\sqrt{(\tan^{-1}(3x))^2}}$$

13. If $F(x) = \sin(\cot^{-1} x)$, then $F'(1) =$?

$$F'(x) = \frac{d}{dx} \sin(\cot^{-1} x) = \cos(\cot^{-1} x) \cdot \frac{d}{dx}(\cot^{-1} x) = \cos(\cot^{-1} x) \cdot \frac{-1}{x^2+1} = \frac{-\cos(\cot^{-1} x)}{x^2+1}$$

Let $\theta = \cot^{-1} x$. Then we want to know what $\cos \theta$ equals, given that $\cot \theta = x$. Draw a simple right triangle with an angle $\theta$ such that $\cot \theta = x$:

This triangle is the right one:

$$\cot \theta = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{x}{1} = x$$

Fill in the missing leg:

Now we see that $\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2+1}}$. Thus we obtain:
\[ F'(x) = \frac{-\cos(\cot^{-1} x)}{x^2 + 1} = \frac{-x}{\sqrt{x^2 + 1}} = \frac{-x}{(x^2 + 1)^{\frac{3}{2}}} \]

\[ F'(1) = \frac{-1}{(1^2 + 1)^{\frac{3}{2}}} = \frac{-1}{2\sqrt{2}} \]

14. If \( f(x) = \sin(x + \sec^{-1} x) \), then \( f'(2) = ? \)

\[ f'(x) = \frac{d}{dx} \sin(x + \sec^{-1} x) = \cos(x + \sec^{-1} x) \frac{dx}{dx}(x + \sec^{-1} x) = \cos(x + \sec^{-1} x) \left(1 + \frac{1}{x\sqrt{x^2 - 1}}\right) \]

\[ = \frac{(1 + x\sqrt{x^2 - 1})\cos(x + \sec^{-1} x)}{x\sqrt{x^2 - 1}} \]

\[ f'(2) = \frac{(1 + 2\sqrt{2^2 - 1})\cos(2 + \sec^{-1} 2)}{2\sqrt{2^2 - 1}} = \frac{(1 + 2\sqrt{3}) \cos \left(2 + \frac{\pi}{3}\right)}{2\sqrt{3}} = \frac{(1 + 2\sqrt{3}) \cos \left(\frac{6 + \pi}{3}\right)}{2\sqrt{3}} \]

15. If \( G(x) = \cot^{-1}(x + \csc^{-1} x) \), then \( G'(2) = ? \)

\[ G'(x) = \frac{d}{dx} \cot^{-1}(x + \csc^{-1} x) = \frac{-1}{(x + \csc^{-1} x)^2 + 1} \left(1 + \frac{-1}{x\sqrt{x^2 - 1}}\right) \]

\[ G'(2) = \frac{1 - 2\sqrt{2^2 - 1}}{2\sqrt{2^2 - 1} \left(\left(2 + \csc^{-1} 2\right)^2 + 1\right)} = \frac{1 - 2\sqrt{3}}{2\sqrt{3}} \left(\left(2 + \frac{\pi}{6}\right)^2 + 1\right) = \frac{1 - 2\sqrt{3}}{2\sqrt{3}} \left(\left(\frac{12 + \pi}{36}\right)^2 + 1\right) = \frac{18(1 - 2\sqrt{3})}{\sqrt{3}(12 + \pi)^2 + 36} \]

16. If \( y = \tan^{-1}\left(\tan^{-1}(\tan^{-1} x)\right) \), find \( \frac{dy}{dx} \).

\[ \frac{dy}{dx} = \frac{d}{dx} \tan^{-1}\left(\tan^{-1}(\tan^{-1} x)\right) = \frac{d}{dx} \left(\tan^{-1}(\tan^{-1} x)\right) + \frac{d}{dx} \left(\tan^{-1}(\tan^{-1} x)\right) = \frac{\frac{d}{dx} \tan^{-1} x}{\left(\tan^{-1}(\tan^{-1} x)\right)^2 + 1} \]
\[
\frac{dy}{dx} = \frac{1}{x^2 + 1} \left( \frac{\tan^{-1}(x)}{\tan^{-1}(\tan^{-1}x)} \right)^2 + 1 = \frac{1}{(x^2 + 1)\left(\left(\tan^{-1}(x)\right)^2 + 1\right)\left(\left(\tan^{-1}(\tan^{-1}x)\right)^2 + 1\right)}
\]

**Integral Examples**

17. \( \int \frac{\cos x \, dx}{(\sin x)^2 + 1} = ? \) Let \( u = \sin x \Rightarrow du = \cos x \, dx \).

Then \( \int \frac{\cos x \, dx}{(\sin x)^2 + 1} = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C = \tan^{-1}(\sin x) + C \)

18. \( \int \frac{\sec^2 x \, dx}{\sqrt{1 - \tan^2 x}} = ? \) Let \( u = \tan x \Rightarrow du = \sec^2 x \, dx \)

Then \( \int \frac{\sec^2 x \, dx}{\sqrt{1 - \tan^2 x}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1}(\tan x) + C \)

19. \( \int \frac{3x^2 - 10x}{(x^3 - 5x^2 + 1)^2 + 1} \, dx = ? \) Let \( u = x^3 - 5x^2 + 1 \Rightarrow du = (3x^2 - 10x) \, dx \).

Then \( \int \frac{3x^2 - 10x}{(x^3 - 5x^2 + 1)^2 + 1} \, dx = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C = \tan^{-1}(x^3 - 5x^2 + 1) + C \)

20. \( \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x \, dx}{\tan x \sqrt{\tan^2 x - 1}} = ? \) Let \( u = \tan x \Rightarrow du = \sec^2 x \, dx \).

Also, \( x = \frac{\pi}{6} \Rightarrow u = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \) & \( x = \frac{\pi}{3} \Rightarrow u = \tan \frac{\pi}{3} = \sqrt{3} \), so

\[ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2 x \, dx}{\tan x \sqrt{\tan^2 x - 1}} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{du}{u \sqrt{u^2 - 1}} = \sec^{-1} u \Bigg|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \sec^{-1}(\sqrt{3}) - \sec^{-1}\left(\frac{1}{\sqrt{3}}\right) \]
The boundaries in the following picture are graphs of all of the trigonometric functions and their inverse functions, over the rectangle $[0, \pi] \times [0, \pi]$, reflected through the $x$ and $y$ axes:

By Rafael Espericueta, 2009