6.1  Exponential and Logarithmic Functions (an overview)

This section reviews the exponential and logarithmic functions. These functions are inverse functions of each other.

Exponential and logarithmic functions are all around us, manifesting in diverse phenomena. Exponential functions are involved in the rate of growth of bacteria in potato salad, radioactive decay, memory loss and many other situations. Logarithmic functions are present whenever a cup of soup cools off (or warms up), and are even involved in every sensation you’ve ever experienced!

Exponential Functions

Exponential functions are used to model population growth, radioactive decay, compound interest, and many other practical situations.

Polynomials involve terms with variables raised to constant powers, like \( x^8 \). Exponentials involve terms with positive constants raised to variable powers, like \( 8^x \). Despite their similar appearance, these are fundamentally quite different kinds of functions.

To the right is the graph of the exponential function \( f(x) = 2^x \).

Notice that each time \( x \) increases by one, the function doubles. Exponentials grow faster than any polynomial!

Increasing exponential functions of this type can be used to model population growth and compound interest. Exponential functions arise anytime a quantity is directly proportional to the rate of change of that quantity.

The above is an example of an increasing exponential function. Next we’ll look at a decreasing exponential function.

To the right is the graph of the exponential function \( g(x) = 2^{-x} \).

Notice that each time \( x \) increases by one, the function decreases by half.
The above decreasing exponential can be either thought of as a number greater than one raised to a negative exponent, or as a number less than one raised to a positive exponent. This is implied by the properties of exponents. Decreasing exponential functions are used to model radioactive decay.

One very practical application of exponential functions is the calculation of compound interest. Usually interest is compounded monthly. We’ll let \( A(t) \) denote the amount of money the account has grown to after \( t \) years, assuming it’s compounded \( N \) times per year at annual interest rate \( r \). Then

\[
A(t) = A_0 \left(1 + \frac{r}{N}\right)^{Nt}.
\]

Continuous compounding of interest is obtained by taking the limit as the number of compoundings per year, \( N \), approaches infinity. Later in this chapter you’ll see that how the equation for continuous compounding of interest, with initial amount \( A_0 \) and annual interest rate \( r \), is given by:

\[
A(t) = A_0 e^{rt}.
\]

Unconstrained population growth is modeled by essentially the same equation

\[
P(t) = P_0 e^{rt},
\]

where \( P(t) \) is the population after \( t \) time units (usually minutes or years), given an initial population \( P_0 \). The constant \( r \) is the growth rate per time unit; the larger \( r \), the faster the population grows. The constant \( e \approx 2.718281828459 \) is a special irrational constant, related to \( \pi \). Exponentials of this number (Euler’s constant) are built-in functions of all scientific calculators, which makes such exponentials easy to calculate.

Radioactive decay is governed by the equation

\[
A(t) = A_0 e^{-rt},
\]

where \( A(t) \) is the amount of a radioactive substance remaining after \( t \) years, given an initial amount \( A_0 \) of the substance. The constant \( r \) depends on the radioactive substance (it’s larger, the faster the substance decays).

**Logarithmic Functions**

Exponential functions are one-to-one, and so have inverse functions. The inverse exponential functions are called *logarithmic functions*. For example, the inverse of the function

\[
y = 5^x
\]

is the logarithmic function
\[ y = \log_5 x. \]

"the log to the base five of \( x \)"

Since the graph of an inverse function is but the graph of the function reflected through the line \( y = x \), we know what the graph of this logarithmic function looks like:

Notice from the graph that logarithms have only positive numbers in their domains, just as exponentials only have positive numbers in their ranges.

Logarithmic functions have many practical uses, from solving exponential equations to understanding how our senses work. Even the cooling of a cup of coffee can be modeled using logarithmic functions.

Since logarithmic functions and exponential functions are inverse functions of each other, it follows that

\[ \log_a (a^x) = x, \]

and that

\[ a^{\log_a x} = x. \]

(Remember, functions and inverse functions undo each other!)

One useful way of thinking about exponentials and logarithms is that one may write the exponential equation, \( A = B^C \), in the equivalent logarithmic form, \( \log_B A = C \).

Logarithms are used to get at the exponents when solving equations. Some exponential equations can be solved by simply writing them in logarithmic form. For example, to solve the equation \( 10^x = 17 \), simply write it in logarithmic form, obtaining \( x = \log_{10} 17 \).

Scientific calculators have a built-in \( \log_{10} \) function, usually just written \( \log \). Base 10 logs are called common logarithms. Using a calculator we can obtain a numerical approximation to our solution for \( x \):

\[ x = \log 17 \approx 1.23044892138. \]
Some logarithmic equations can be solved by writing them in exponential form. For example, to solve the equation \( \log_2 x = 3 \), simply write it in exponential form, \( x = 2^3 = 8 \).

**Properties of Logarithms**

All the properties of logs come from the fact that they are inverse exponential functions. Many properties can be instantly found by going from a known exponential equation to its logarithmic form. The following three properties are based on the equivalence:

\[
\begin{align*}
\log_a c &= b \\
\iff
a^b &= c
\end{align*}
\]

<table>
<thead>
<tr>
<th>Property of Exponents</th>
<th>Property of Logarithms</th>
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<tbody>
<tr>
<td>( a^0 = 1 )</td>
<td>( \log_a 1 = 0 )</td>
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<tr>
<td>( a^1 = a )</td>
<td>( \log_a (a) = 1 )</td>
</tr>
<tr>
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<td>( \log_a (a^n) = n )</td>
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Other very useful properties of logarithms are the following product, quotient, and power rules, as well as the change of base formula that we use to convert from one logarithmic base to another:

- **Product rule:** \( \log_a (bc) = \log_a (b) + \log_a (c) \)
- **Quotient rule:** \( \log_a \left(\frac{b}{c}\right) = \log_a (b) - \log_a (c) \)
- **Power rule:** \( \log_a (b^n) = n \cdot \log_a (b) \)
- **Change of base formula:** \( \log_a w = \frac{\log_b w}{\log_b a} \)

The change of base formula is often used to calculate arbitrary logarithms using either base 10 logarithms (common logs) or base \( e \) logarithms (natural logs):

\[
\log_a w = \frac{\log w}{\log a} = \frac{\ln w}{\ln a}
\]

Since \( a = b^{\log_b(a)} \), we can raise both sides to the \( x \) power to get: \( a^x = \left(b^{\log_b(a)}\right)^x \), and so:

\[
a^x = b^{x \log_b(a)}
\]

This is a change of base formula for exponential functions. In particular \( a^x = 10^{x \log_{10} a} = e^{x \ln a} \).
Now Add a Pinch of Calculus…

Let \( f(x) = a^x \), and let’s try to compute this function’s derivative. By definition,

\[
\frac{d}{dx} a^x = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}
\]

If we assume that \( \lim_{h \to 0} \frac{a^h - 1}{h} = C \) for some constant \( C \), (don’t worry, we’ll deal with all the gory details in the next sections!), the above becomes:

\[
\frac{d}{dx} a^x = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot C = Ca^x
\]

“The rate of change of \( f \) is directly proportional to \( f \).”

Next we’ll let \( g(x) = \log_a(x) \), and try to find its derivative. But first, recall the standard process for finding the derivative of an inverse function:

\[
(f^{-1}(x))' = \frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x = 1
\]

[ definition of inverse function ]

\[
(f^{-1}(x))' \cdot \frac{d}{dx} f^{-1}(x) = 1
\]

[ differentiate each side of the equation ]

\[
(f^{-1}(x))' = \frac{1}{\frac{d}{dx} f^{-1}(x)}
\]

[ chain rule ]

Applying this with \( f(x) = a^x \) and \( f^{-1}(x) = \log_a(x) \):

\[
\frac{d}{dx} \log_a(x) = \frac{1}{x \ln(a)}
\]
\[ a^{\log_a(x)} = x \quad \text{[ definition of inverse function]} \]
\[ \frac{d}{dx} a^{\log_a(x)} = \frac{d}{dx} x \quad \text{[ differentiate each side of the equation]} \]
\[ C a^{\log_a(x)} \cdot \frac{d}{dx} \log_a(x) = 1 \quad \text{[ chain rule, and new derivative rule for exponentials]} \]
\[ C x \cdot \frac{d}{dx} \log_a(x) = 1 \quad \text{[ logs and exponentials are inverse functions]} \]

\[
\frac{d}{dx} \log_a(x) = \frac{1}{C} \cdot \frac{1}{x} = \frac{1/C}{x}
\]

“The rate of change of \( y = \log_a(x) \) is inversely proportional to \( x \)”

Example:

Suppose that \( f \) is invertible and differentiable everywhere, and that \( f^{-1}(2) = 1 \).

Suppose further that \( f'(1) = 7 \). Then what is the derivative of \( f^{-1} \), evaluated at \( x = 2 \)?

\[ f\left(f^{-1}(x)\right) = x \quad \text{[ definition of inverse function]} \]
\[ \frac{d}{dx} f\left(f^{-1}(x)\right) = \frac{d}{dx} x \quad \text{[ differentiate each side of the equation]} \]
\[ f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1 \quad \text{[ chain rule]} \]
\[ \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \]

Thus,
\[ \frac{d}{dx} f^{-1}(x) \bigg|_{x=2} = \frac{1}{f'(f^{-1}(2))} = \frac{1}{7} \]