6.2 The Exponential $e^x$ and the Natural Logarithm $\ln x$

In the last section, we attempted to take the derivative of the exponential function $f(x) = a^x$, using the definition of the derivative as a limit of a difference quotient. We found that

$$f'(x) = \frac{d}{dx} a^x = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot C,$$

Where $\lim_{h \to 0} \frac{a^h - 1}{h} = C$ for some constant $C$. Then $f'(0) = a^0 \cdot C = 1 \cdot C = C$.

We also found the derivative of the logarithmic function $g(x) = \log_a(x)$ to be

$$g'(x) = \frac{d}{dx} \log_a(x) = \frac{1}{Cx},$$

with that same constant $C$ that appeared when we took the derivative of the related exponential function, $a^x$. Then we also have $g'(1) = \frac{1}{C \cdot 1} = \frac{1}{C}$.

Let’s try this derivative, $g'(1)$, once more, using the definition of derivative:

$$g'(1) = \left( \frac{d}{dx} \log_a(x) \right) \bigg|_{x=1} = \lim_{h \to 0} \frac{\log_a(1+h) - \log_a(1)}{h} = \lim_{h \to 0} \frac{\log_a(1+h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\log_a(1+h)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \log_a(1+h)$$

$$= \lim_{h \to 0} \log_a\left(1 + \frac{1}{n}\right)^{\frac{1}{h}}$$

Next we invoke a standard trick. Make the substitution $n = \frac{1}{h}$, so also $h = \frac{1}{n}$. Then as $h$ goes to 0, $n$ goes to infinity, and vice-versa (well, technically, $h$ has to approach 0 from the right). Making this substitution, we get:

$$\lim_{h \to 0} \log_a\left(1 + \frac{1}{n}\right)^{\frac{1}{h}} = \lim_{n \to \infty} \log_a\left(1 + \frac{1}{n}\right)^n$$

If we assume that the log is continuous at the limit, the above becomes:
\[
\lim_{n \to \infty} \log_a \left( 1 + \frac{1}{n} \right)^n = \log_a \left( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \right)
\]

The limit \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \), is one way to define the Euler number, \( e \):

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e
\]

This special transcendental number (like \( \pi \)) arises frequently in higher math, and is approximately:

\[
e \approx 2.71828182845904523536028747135266249775725170937
\]

Logarithms to this base are called “natural logarithms”, and we write \( \log_e x = \ln x \).

Assuming this definition for \( e \), we obtain:

\[
g'(1) = \left( \frac{d}{dx} \log_a(x) \right) \bigg|_{x=1} = \log_a(e)
\]

Above we earlier found that \( g'(x) = \frac{1}{Cx} \), hence \( g'(1) = \frac{1}{C} \). And so we get that

\[
\log_a(e) = \frac{1}{C} \quad \Rightarrow \quad C = \frac{1}{\log_a(e)}.
\]

But we can simplify this further. Use the logarithm change of base formula, and the above becomes:

\[
C = \frac{1}{\log_a(e)} = \frac{1}{\left( \frac{\ln e}{\ln a} \right)} = \frac{\ln a}{\ln e} = \frac{\ln a}{1} = \ln a
\]

We’ve solved for that mysterious constant, \( C \)! Putting this new information together with our earlier results, we get:

\[
\frac{d}{dx} a^x = a^x \cdot \ln a \quad \& \quad \frac{d}{dx} \log_a(x) = \frac{1}{x \cdot \ln(a)}
\]
Those are important results, that we can use to differentiate exponential functions without having to take limits. Oh, and if you look back, you’ll see we’ve also discovered that \( \lim_{h \to 0} \frac{a^h - 1}{h} = \ln a \).

Applying the above derivative formulas in the case where \( a = e \), we get:

\[
\frac{d}{dx} e^x = e^x \quad \text{and} \quad \frac{d}{dx} \ln x = \frac{1}{x}
\]

In base \( e \), calculus is very simple. Perhaps “\( e \)” really stands for “easy”! 😊

One can always convert any other base to the base \( e \) first, and then use the above simple derivative rules, for \( a^x = (e^{\ln a})^x = e^{x \ln a} \), and also \( \log_a x = \frac{\ln x}{\ln a} = \left( \frac{1}{\ln a} \right) \cdot \ln x \). Or just use the previous more complicated formulas, if you prefer (but you’ll have to remember them in that case).

Next, using the chain rule, we can generalize the above derivative formulas:

\[
\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)
\]

Also, \( \frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) \), and so

\[
\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}.
\]

That last expression, \( \frac{f'(x)}{f(x)} \) is called the relative rate of change of \( f \).

The derivative of the natural log of a function equals the relative rate of change of the function.

To understand the meaning of the relative rate of change of a function, imagine that Bill Gates won \$1000000\) in a lottery. His \( f' \) (the change in his fortune) would equal \$1000000\), which equals my \( f' \) should I have won that amount. But for Bill, this would not even faze him, whereas I might well jump for joy. The derivative (rate of change) doesn’t capture my joy and Bill’s lack of it, but the relative rate of change does! For Bill’s relative rate of change would be something like

\[
\frac{f'}{f} = \frac{1000000}{50000000000} = \frac{1}{50000},
\]

not much, whereas my relative rate of change would be more like

\[
\frac{f'}{f} = \frac{1000000}{5000} = 200,
\]

a much larger amount that could make me jump for joy!
Note that from the differentiation formula \( \frac{d}{dx} e^x = e^x \) we can derive the differentiation formula for the natural log, using the same inverse function derivative technique we used earlier in the course to compute the derivatives of the inverse trigonometric functions:

\[
e^{\ln x} = x
\]

\[
\frac{d}{dx} e^{\ln x} = 1
\]

\[
e^{\ln x} \cdot \frac{d}{dx} \ln x = 1
\]

\[
x \cdot \frac{d}{dx} \ln x = 1
\]

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

Also, from the differentiation formula \( \frac{d}{dx} \ln x = \frac{1}{x} \) we can derive the differentiation formula for the exponential function, using the same inverse function derivative technique:

\[
\ln \left( e^x \right) = x
\]

\[
\frac{d}{dx} \ln \left( e^x \right) = 1
\]

\[
\frac{1}{e^x} \frac{d}{dx} e^x = 1
\]

\[
\frac{d}{dx} e^x = e^x
\]

**Differentiation Examples**

The following examples make use of these derivative formulas we’ve just derived:

<table>
<thead>
<tr>
<th>Derivative Formula</th>
<th>Again ( simpler notation )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x) )</td>
<td>( \frac{d}{dx} e^u = e^u \cdot u' )</td>
</tr>
<tr>
<td>( \frac{d}{dx} \ln \left( f(x) \right) = \frac{f'(x)}{f(x)} )</td>
<td>( \frac{d}{dx} \ln u = \frac{u'}{u} )</td>
</tr>
</tbody>
</table>
1. \[
\frac{d}{dx} e^{4x} = e^{4x} \cdot \frac{d}{dx} (4x) = 4e^{4x}
\]

2. \[
\frac{d}{dx} e^{\sin(x)} = e^{\sin(x)} \cdot \frac{d}{dx} \sin(x) = e^{\sin(x)} \cos(x)
\]

3. \[
\frac{d}{dx} \ln(x^2) = \frac{\frac{d}{dx} x^2}{x^2} = \frac{2x}{x^2} = \frac{2}{x}
\]

4. \[
\frac{d}{dx} \ln(\cos x) = \frac{\frac{d}{dx} \cos x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x
\]

For the following examples, use the fact that \(a^x = e^{x \ln a}\).

5. \[
\frac{d}{dx} 7^{3x} = \frac{d}{dx} e^{3x \ln 7} = e^{3x \ln 7} \cdot \frac{d}{dx} (3x \ln 7) = 7^{3x} \cdot (3 \ln 7) = (3 \ln 7) \cdot 7^{3x}
\]

6. \[
\frac{d}{dx} 3^{\tan(x)} = \frac{d}{dx} e^{\tan(x) \ln 3} = e^{\tan(x) \ln 3} \cdot \frac{d}{dx} (\tan(x) \ln 3)
\]
\[
= 3^{\tan(x)} \sec^2(x)(\ln 3) = (\ln 3) \cdot \sec^2(x) \cdot 3^{\tan(x)}
\]

For the following examples, use the logarithm change of base formula, \(\log_a x = \frac{\ln x}{\ln a}\).

7. \[
\frac{d}{dx} \log_2(x^5) = \frac{d}{dx} \frac{\ln(x^5)}{\ln 2} = \frac{1}{\ln 2} \cdot \frac{d}{dx} \ln(x^5) = \frac{1}{\ln 2} \cdot \frac{d}{dx} x^5
\]
\[
= \frac{1}{\ln 2} \cdot 5x^4 = \frac{5}{(\ln 2)x}
\]

8. \[
\frac{d}{dx} \log_{11}(x^3 - e^x) = \frac{d}{dx} \frac{\ln(x^3 - e^x)}{\ln(11)} = \frac{1}{\ln(11)} \cdot \frac{d}{dx} \ln(x^3 - e^x)
\]
\[
\frac{d}{dx} \left( x^3 - e^x \right) = \frac{3x^2 - e^x}{x^3 - e^x}
\]

Whenever you find a new derivative formula, just “flip it over” and you’ll find a new integral formula.

The derivative formulas \( \frac{d}{dx} e^u = e^u \cdot u' \) \& \( \frac{d}{dx} \ln u = \frac{u'}{u} \), yield the following integral formulas:

\[
\left[ \begin{array}{c}
\int e^{f(x)} \cdot f'(x) \, dx = e^{f(x)} + C \\
\int \frac{f'(x)}{f(x)} \, dx = \ln(f(x)) + C
\end{array} \right]
\]

There are two very important special cases. Since \( \frac{d}{dx} e^x = e^x \), we must also have that

\[
\int e^x \, dx = e^x + C.
\]

And since \( \frac{d}{dx} \ln x = \frac{1}{x} \), we must also have that

\[
\int \frac{1}{x} \, dx = \ln x + C.
\]

It’s easy to convert one of the more complicate integral formulas above \([*]\) to one of these simpler formulas via the substitution \( u = f(x), \ du = f'(x) \, dx \).

That last integral formula fills a gap. Recall the integral power rule,

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for } n \neq -1
\]

The case where \( n = -1 \) we couldn’t deal with until now. But now we know that

\[
\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \int \frac{dx}{x} = \ln x + C
\]

We’ll come back again to integrals involving the natural logarithm in section 6.4.
Integration Examples

1. \( \int xe^{x^2} \, dx = ? \)
   Let \( u = x^2 \Rightarrow du = 2x \, dx \Rightarrow \frac{1}{2} \, du = x \, dx \). Then,
   \[
   \int xe^{x^2} \, dx = \frac{1}{2} \int e^u \, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C
   \]

2. \( \int_{0}^{\pi/6} e^{\cos(3x)} \sin(3x) \, dx = ? \)
   Let \( u = \cos(3x) \Rightarrow du = -\sin(3x) \cdot 3 \, dx \Rightarrow \frac{1}{3} \, du = \sin(3x) \, dx \).
   
   \[
   \int_{0}^{\pi/6} e^{\cos(3x)} \sin(3x) \, dx = \frac{1}{3} \int_{1}^{0} e^u \, du = -\frac{1}{3}e^u \bigg|_{1}^{0} = -\frac{1}{3}(e^0 - e^1) = -\frac{1}{3}(1 - e) = \frac{e-1}{3}
   \]

   For the following examples, we use the fact that \( a^x = (e^{\ln a})^x = e^{x \ln a} \).

3. \( \int 10^x \, dx = ? \)
   Since \( 10^x = e^{x \ln 10} \), we have
   \[
   \int 10^x \, dx = \int e^{x \ln 10} \, dx
   \]
   Let \( u = x \ln 10 \Rightarrow du = (\ln 10) \, dx \Rightarrow \frac{1}{\ln 10} \, du = dx \). Then,
   \[
   \int 10^x \, dx = \int e^{x \ln 10} \, dx = \frac{1}{\ln 10} \int e^u \, du = \frac{1}{\ln 10} \cdot e^u + C = \frac{e^{x \ln 10}}{\ln 10} + C = \frac{10^x}{\ln 10} + C
   \]

4. Suppose that \( f'(x) = \csc^2(x) \cdot 3 \cot x \), and that \( f\left(\frac{\pi}{2}\right) = 0 \).
   Then \( f\left(\frac{\pi}{4}\right) = ? \)
By the Fundamental Theorem of Calculus,

\[ f(x) = \int f'(x)\,dx = \int \csc^2(x) \cdot 3^{\cot x} \,dx \]

Let \( u = \cot x \) \( \Rightarrow \) \( du = -\csc^2 x \,dx \) \( \Rightarrow \) \( -du = \csc^2 x \,dx \). Then,

\[ f(x) = \int \csc^2(x) \cdot 3^{\cot(x)} \,dx = -\int 3^u \,du = -\int e^{u\ln 3} \,du \]

Next let \( v = (\ln 3)u \) \( \Rightarrow \) \( dv = (\ln 3) \,du \) \( \Rightarrow \) \( \frac{1}{\ln 3}dv = du \). Then,

\[ f(x) = -\int e^{u\ln 3} \,du = -\frac{1}{\ln 3} \int e^v \,dv = -\frac{1}{\ln 3} \cdot e^v + C = -\frac{1}{\ln 3} \cdot e^{u\ln 3} + C \]

\[ = -\frac{1}{\ln 3} \cdot 3^u + C = -\frac{1}{\ln 3} \cdot 3^{\cot x} + C = \frac{3^{\cot x}}{\ln 3} + C \]

We are given that \( f'(\frac{\pi}{2}) = 1 \), so we can solve for the constant of integration \( C \):

\[ f\left(\frac{\pi}{2}\right) = -3^{\cot \frac{\pi}{2}} + C = 0 \quad \Rightarrow \quad -\frac{3^0}{\ln 3} + C = 0 \]

\[ \Rightarrow \quad -\frac{1}{\ln 3} + C = 0 \quad \Rightarrow \quad C = \frac{1}{\ln 3} \]

Thus,

\[ f(x) = -\frac{3^{\cot x}}{\ln 3} + \frac{1}{\ln 3} = \frac{1 - 3^{\cot x}}{\ln 3}, \]

whence

\[ f\left(\frac{\pi}{4}\right) = \frac{1 - 3^{\cot \frac{\pi}{4}}}{\ln 3} = \frac{1 - 3^1}{\ln 3} = \frac{2}{\ln 3} \]