7.2 Trigonometric Integrals

In this section, we’ll study techniques for integrating products of some of the trigonometric functions. The two main approaches to consider are integration by parts (or better yet its more streamlined version, tabular integration), and techniques that make use of the trigonometric identities. We’ll begin with the trigonometric identity approach.

Integrals of Products of Powers of Sines and Cosines \[ \int \sin^m x \cdot \cos^n x \, dx \]

The first type of integrals we’ll consider are of the form \[ \int \sin^m x \cdot \cos^n x \, dx \], where \( m \) and \( n \) are positive integers, and at least one of them is odd (one of them may be 0). The idea is to convert everything to be all sines or all cosines, except for one lone factor equal to the other trig function.

Example 1. \[ \int \sin^2 x \cdot \cos^3 x \, dx = ? \]

\[ \int \sin^2 x \cdot \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx \]

Next, rewrite the \( \cos^2 x \) in terms of the sine function, using the Pythagorean identity, \( \cos^2 x = 1 - \sin^2 x : \)

\[ = \int \sin^2 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx \]

Finally, we can make the substitution \( u = \sin x, \ du = \cos x \, dx \), and our integral becomes:

\[ \int u^2 (1-u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C \]

\[ \Rightarrow \int \sin^2 x \cdot \cos^3 x \, dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \]

Example 2. \[ \int \sin^5 x \cdot \cos^4 x \, dx = ? \]

\[ = \int \sin^4 x \cdot \sin x \cdot \cos^4 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \sin x \, dx \]

Next, rewrite the \( \sin^4 x = (\sin^2 x)^2 \) in terms of the cosine function, using the Pythagorean identity, \( \sin^2 x = 1 - \cos^2 x : \)

\[ = \int (\sin^2 x)^2 \cdot \cos^4 x \cdot \sin x \, dx = \int (1 - \cos^2 x)^2 \cdot \cos^4 x \cdot \sin x \, dx \]

Finally, we can make the substitution \( u = \cos x, \ du = -\sin x \, dx \), and our integral becomes:

\[ = -\int (1-u^2)^2 u^4 \, du = -\int (1-2u^2+u^4) u^4 \, du = \int (-u^4 + 2u^6 - u^8) \, du = -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C \]

\[ \Rightarrow \int \sin^5 x \cdot \cos^4 x \, dx = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + C \]
These types of integrals can also be solved using tabular integration. To compare the approaches, we’ll now redo the last problem, using tabular integration. For ease of writing, we’ll let \( s = \sin x \) and \( c = \cos x \):

\[
\int \sin^5 x \cdot \cos^4 x \, dx = \int s^5 c^4 \, dx = ?
\]

| \( + \) | \( s^4 \) | \( c^4 s \) |
| \( - \) | \( 4s^3 c \) | \( - \frac{c^5}{5} \) |
| \( - \) | \( 4s^2 \) | \( - \frac{c^6 s}{5} \) |
| \( + \) | \( 8sc \) | \( \frac{c^7}{5} \cdot 7 \) |

Notice how we “borrow” an \( s \) from the first column entry so we could integrate the second column entry. We can integrate that last row, so we end our tabular tableau. We have:

\[
\int s^5 c^4 \, dx = -\frac{s^4 c^5}{5} - \frac{4s^2 c^7}{35} + \frac{8}{35} \cdot \int c^8 s \, dx
\]

Letting \( u = \cos x = c \), \( du = -s \, dx \), our integral becomes:

\[
= -\frac{s^4 c^5}{5} - \frac{4s^2 c^7}{35} - \frac{8}{35} \cdot \int u^8 du = -\frac{s^4 c^5}{5} - \frac{4s^2 c^7}{35} - \frac{8}{35} \cdot \frac{u^9}{9} + C = -\frac{s^4 c^5}{5} - \frac{4s^2 c^7}{35} - \frac{8}{35} \cdot \frac{c^9}{9} + C
\]

\[
\Rightarrow \int \sin^5 x \cdot \cos^4 x \, dx = -\frac{\sin^4 x \cos^5 x}{5} - \frac{4\sin^2 x \cos^7 x}{35} - \frac{8\cos^9 x}{315} + C
\]

This answer looks quite different from our previous solution. Unfortunately, this happens frequently with trigonometric expressions, since trigonometry is so rich in identities. But if you express our last solution in terms of the cosine, and simplify, you do indeed obtain our previous result. Nonetheless, our first solution seems to be less work. So henceforth we usually won’t use tabular integration to solve these types of integrals, though the above example shows that we could.

Example 3. \( \int \sin^3 x \cdot \cos^3 x \, dx = \int \sin^3 x \cdot \cos^2 x \cdot \cos x \, dx \)

Next, rewrite \( \cos^2 x \) in terms of the cosine function, using the Pythagorean identity, \( \cos^2 x = 1 - \sin^2 x \):

\[
= \int \sin^3 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx
\]

Finally, we can make the substitution \( u = \sin x \), \( du = \cos x \, dx \), and our integral becomes:

\[
= \int u^3 (1-u^2) \, du = \int (u^3 - u^5) \, du = \frac{u^4}{4} - \frac{u^6}{6} + C \quad \Rightarrow \int \sin^3 x \cdot \cos^3 x \, dx = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C
\]
Example 4.  \( \int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx \)

Next, rewrite \( \cos^4 x \) in terms of the sine function, using the Pythagorean identity, \( \cos^2 x = 1 - \sin^2 x \):

\[
\begin{align*}
= & \int (\cos^2 x)^2 \cdot \cos x \, dx = \int (1 - \sin^2 x)^2 \cdot \cos x \, dx = \int (1 - 2 \sin^2 x + \sin^4 x) \cdot \cos x \, dx \\
= & \int \cos x \, dx - 2 \int \sin^2 x \cdot \cos x \, dx + \int \sin^4 x \cdot \cos x \, dx
\end{align*}
\]

Now we make the substitution \( u = \sin x \), \( du = \cos x \, dx \); our integrals become:

\[
\begin{align*}
= & \int du - 2 \int u^2 \, du + \int u^4 \, du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\
\Rightarrow \int \cos^5 x \, dx & = \sin x - \frac{2\sin^3 x}{3} - \frac{\sin^5 x}{5} + C
\end{align*}
\]

Example 5.  \( \int \frac{\sin^3 \left( \ln x \right)}{x} \, dx = ? \)  

First substitute \( u = \ln x \), \( du = \frac{dx}{x} \):

\[
\begin{align*}
= & \int \sin^3 u \, du = \int \sin^2 u \cdot \sin u \, du = \int (1 - \cos^2 u) \cdot \sin u \, du \\
= & \int \sin u \, du - \int \cos^2 u \cdot \sin u \, du = -\cos u - \int \cos^2 u \cdot \sin u \, du
\end{align*}
\]

For the remaining integral, let \( v = \cos u \), \( dv = -\sin u \, du \); our integral then becomes:

\[
\begin{align*}
= & -\cos u + \int v^2 \, dv = -\cos u + \frac{v^3}{3} + C = -\cos u + \frac{\cos^3 u}{3} + C \\
\Rightarrow \int \frac{\sin^3 \left( \ln x \right)}{x} \, dx & = -\cos \left( \ln x \right) + \frac{\cos^3 \left( \ln x \right)}{3} + C
\end{align*}
\]

The next type of integrals we’ll consider are of the form \( \int \sin^m x \cdot \cos^n x \, dx \), where both of the exponents are even. The simplest such integrals you’ve probably seen before:

Example 6.  \( \int \cos^2 x \, dx = ? \)

Use the trigonometric half-angle identity, \( \cos^2 x = \frac{1 + \cos(2x)}{2} \), to rewrite the integrand:

\[
\begin{align*}
\int \cos^2 x \, dx & = \int \frac{1 + \cos(2x)}{2} \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx \\
= & \frac{1}{2} \cdot x + \frac{1}{2} \cdot \sin \frac{2x}{2} + C = \frac{2x + \sin 2x}{4} + C
\end{align*}
\]
Example 7. \[ \int \sin^2 x \, dx = ? \] Use the trig half-angle identity, \( \sin^2 x = \frac{1 - \cos(2x)}{2} \):

\[
\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx \\
= \frac{1}{2} x - \frac{1}{2} \sin 2x + C = \frac{2x - \sin 2x}{4} + C
\]

Those last two examples are the key to integrating all integrals of the form \( \int \sin^m x \cdot \cos^n x \, dx \), where both of the exponents are even. We keep using the same half-angle identities to cut the exponents in half, until we finally get something we can easily integrate.

Example 8. \[ \int \sin^4 x \, dx = \int \left( \sin^2 x \right)^2 dx \] Use the trig half-angle identity, \( \sin^2 x = \frac{1 - \cos(2x)}{2} \):

\[
\int \left( \frac{1 - \cos 2x}{2} \right)^2 dx = \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx \\
= \frac{1}{4} \int dx - \frac{1}{4} \cdot 2 \int \cos 2x \, dx + \frac{1}{4} \int \cos^2 2x \, dx = \frac{1}{4} \cdot x - \frac{1}{2} \cdot \int \cos 2x \, dx + \frac{1}{4} \cdot \int \cos^2 2x \, dx
\]

Notice that for the integral on the far right, we need to invoke the half-angle identity once again.

In this case it takes the form \( \cos^2 (2x) = \frac{1 + \cos(4x)}{2} \), and our integral becomes:

\[
= \frac{x}{4} - \frac{1}{2} \cdot \frac{\sin 2x}{2} + \frac{1}{4} \cdot \int \frac{1 + \cos 4x}{2} \, dx = \frac{x}{4} - \frac{\sin 2x}{2} + \frac{1}{8} \left( \int dx + \int \cos 4x \, dx \right) \\
= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{8} \left( x + \frac{\sin 4x}{4} \right) + C = \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C
\]

\( \Rightarrow \int \sin^4 x \, dx = \frac{12x - 8 \sin 2x + \sin 4x}{32} + C \)

Example 9. \[ \int (\sin x \cos x)^4 \, dx = ? \] ( !Warning! - Not for the feint of heart! )

\[
= \int \left( \sin^2 x \cos^2 x \right)^2 dx = \int \left( \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} \right)^2 dx = \frac{1}{16} \cdot \int (1 - \cos^2 2x)^2 \, dx \\
= \frac{1}{16} \cdot \int (1 - 2 \cos^2 2x + \cos^4 2x) \, dx = \frac{1}{16} \cdot \int dx - \frac{2}{16} \cdot \int \cos^2 2x \, dx + \frac{1}{16} \cdot \int \cos^4 2x \, dx \\
= \frac{1}{16} \cdot x - \frac{1}{8} \cdot \int \cos^2 2x \, dx + \frac{1}{16} \cdot \int \left( \cos^2 2x \right)^2 dx
\]
$$= \frac{x}{16} \frac{1}{8} \int \frac{1 + \cos 4x}{2} \, dx + \frac{1}{16} \int \left( \frac{1 + \cos 4x}{2} \right)^2 \, dx$$

$$= \frac{x}{16} - \frac{1}{16} \cdot \int (1 + \cos 4x) \, dx + \frac{1}{64} \cdot \int (1 + 2 \cos 4x + \cos^2 4x) \, dx$$

$$= \frac{x}{16} - \frac{1}{16} \cdot \int \cos 4x \, dx + \frac{1}{64} \cdot \int \cos 4x \, dx + \frac{1}{32} \cdot \int \cos 4x \, dx + \frac{1}{64} \cdot \int \cos^3 4x \, dx$$

$$= \frac{x}{16} - \frac{x}{16} + \frac{x}{64} + \left( \frac{1}{16} + \frac{1}{32} \right) \cdot \int \cos 4x \, dx + \frac{1}{64} \cdot \int \left( \frac{1 + \cos 8x}{2} \right) \, dx$$

$$= \frac{x}{64} - \frac{1}{32} \cdot \frac{\sin 4x}{4} + \frac{1}{128} \cdot \int (1 + \cos 8x) \, dx = \frac{x}{64} - \frac{\sin 4x}{128} + \frac{1}{128} \cdot \int \cos 8x \, dx$$

$$= \frac{x}{64} - \frac{\sin 4x}{128} + \frac{x}{128} \cdot \frac{\sin 8x}{8} + C \quad \Rightarrow \quad \int (\sin x \cos x)^4 \, dx = \frac{3x}{128} - \frac{\sin 4x}{128} + \frac{\sin 8x}{1024} + C$$

**Integrals of Products of Powers of Tangents and Secants**

$$\int \tan^m x \cdot \sec^n x \, dx$$

Next we’ll consider integrals of the form $$\int \sin^m x \cdot \cos^n x \, dx$$, as before, but now we’ll allow $$m$$ and $$n$$ to any integer, positive or negative or zero. When at least one of these exponents is negative, the solution to the integral will tend to involve logarithms, as we’ve seen in the case where $$m=1$$ & $$n=-1$$,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + C,$$

as well as the case where $$m=-1$$ & $$n=1$$,

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln|\sin x| + C.$$

When at least one of $$m$$ or $$n$$ is negative, you can think of this as a product of positive powers of the tangent and the secant (or their co-functions); for example, $$\int \sin^3 x \cdot \cos^{-5} x \, dx = \int \tan^3 x \cdot \sec^{-2} x \, dx$$.

We’ll now consider integrals of the form $$\int \tan^m x \cdot \sec^n x \, dx$$. The constants $$m$$ and $$n$$ henceforth will refer to this integral.

Recall the rather bizarre trick we used to solve the case where $$m=0$$ & $$n=1$$,

$$\int \sec x \, dx = \int \frac{\sec x \cdot \sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} \, dx$$

And then we notice that the numerator just happens to be the derivative of the denominator. Let $$u = \sec x + \tan x$$, and our integral becomes:
\[
\int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C
\]
\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

The same bizarre trick is used to integrate the cosecant:
\[
\int \csc x \, dx = -\ln |\csc x + \cot x| + C
\]

Another case we’ve already seen is where \( m = 0 \) & \( n = 2 \):
\[
\int \sec^2 x \, dx = \tan x + C
\]

And similarly for the integral of the squared cosecant:
\[
\int \csc^2 x \, dx = -\cot x + C
\]

For “higher” powers, we’ll make use of the Pythagorean identity \( \tan^2 x + 1 = \sec^2 x \), which as you’ve seen arises from the famous identity \( \sin^2 x + \cos^2 x = 1 \) by dividing both sides by \( \cos^2 x \).

Now for something new, the case where \( m = 0 \) & \( n = 3 \).

Example 10. \( \int \sec^3 x \, dx = ? \) For this one, we’ll start with tabular integration!

| + | \sec x | \sec^2 x |
| - | \sec x \tan x | \tan x |

Thus,
\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx
\]
\[
= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx
\]
\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx
\]
\[
2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx = \sec x \tan x + \ln |\sec x + \tan x|
\]
\[
\Rightarrow \int \sec^3 x \, dx = \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C
\]

The very similar case \( \int \csc^3 x \, dx \), is left for you to do! 😊

Next we’ll solve the case where \( m = 2 \) & \( n = 0 \) :

Example 11. \( \int \tan^2 x \, dx = ? \) (This one’s surprisingly and refreshingly easy!)
\[
\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \int \sec^2 x \, dx - \int dx = \tan x - x + C
\]
Example 12. \[ \int \tan^3 x \, dx = ? \] (This one’s also not too bad, the case \( m = 3 \) & \( n = 0 \).)

\[
\int \tan^3 x \, dx = \int \tan x \cdot \tan^2 x \, dx = \int \tan x \cdot (\sec^2 x - 1) \, dx = \int \tan x \sec^2 x \, dx - \int \tan x \, dx
\]

For the first remaining integral, let \( u = \tan x \), \( du = \sec^2 x \, dx \):

\[
\int \tan^3 x \, dx = \int u \, du + \ln |\cos x| = \frac{u^2}{2} + \ln |\cos x| + C = \frac{\tan^2 x}{2} + \ln |\cos x| + C
\]

Example 13. \[ \int \tan^m x \cdot \sec^2 x \, dx = ? \] (This is the case \( m \) is positive and \( n = 2 \).)

Just let \( u = \tan x \), \( du = \sec^2 x \, dx \):

\[
\int \tan^m x \cdot \sec^2 x \, dx = \int u^m \, du = \frac{u^{m+1}}{m+1} + C = \frac{\tan^{m+1} x}{m+1} + C
\]

Example 14. \[ \int \tan^4 x \sec^4 x \, dx = ? \] (\( m = 4 \& n = 4 \))

\[
\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \, dx = \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx = \int \tan^6 x \sec^2 x \, dx + \int \tan^4 x \sec^2 x \, dx
\]

Now let \( u = \tan x \), \( du = \sec^2 x \, dx \):

\[
\int u^6 \, du + \int u^4 \, du = \frac{u^7}{7} + \frac{u^5}{5} + C = \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C
\]

Example 15. \[ \int \tan^2 x \sec^3 x \, dx = ? \] (\( m = 2 \& n = 3 \))

We’ll try tabular integration on this one; let \( T = \tan x \) and \( S = \sec x \), for less writing. Using these letters, be on the lookout for the identity \( T^2 + 1 = S^2 \), and use the fact that \( T' = S^2 \) and \( S' = ST \):

\[
\begin{array}{ccc}
+ & T & S^2 \cdot ST \\
\downarrow & \downarrow & \downarrow \\
- & S^2 & S^3 \\
\end{array}
\]

Thus,

\[
\int T^2 S^3 \, dx = \frac{TS^3}{3} - \frac{1}{3} \int S^5 \, dx
\]

Often integration by parts (or tabular integration) allow one to convert one case to another. Now we can solve the case \( m=2 \& n=3 \) if we can but solve the case \( m=0 \& n=5 \). So we’ll focus on \( \int S^5 \, dx \), and then we can find our desired integral solution!
\[ \int S^5 \, dx = \int S^3S^2 \, dx = \int S^3 \left(T^2 + 1\right) \, dx = \int S^3T^2 \, dx + \int S^3 \, dx = \int S^3T^2 \, dx + \frac{ST + \ln|S + T|}{2} \]

So now we are done if we can solve the remaining integral, \( \int S^3T^2 \, dx \). But this is the original integral!

\[ \int T^2S^3 \, dx = \frac{TS^3}{3} - \frac{1}{3} \int S^3 \, dx = \frac{TS^3}{3} - \frac{1}{3} \left( \int S^3T^2 \, dx + \frac{ST + \ln|S + T|}{2} \right) \]

\[ \int T^2S^3 \, dx = \frac{TS^3}{3} - \frac{1}{3} \int T^2S^3 \, dx - \frac{1}{3} \frac{ST + \ln|S + T|}{2} \]

\[ \frac{4}{3} \int T^2S^3 \, dx = \frac{TS^3}{3} - \frac{ST + \ln|S + T|}{6} \]

\[ \int T^2S^3 \, dx = \frac{3}{4} \left( \frac{TS^3}{3} - \frac{ST + \ln|S + T|}{6} \right) \]

\[ \int \tan^2 x \sec^3 x \, dx = \frac{\tan x \sec^3 x}{4} - \frac{\tan x \sec x + \ln|\tan x + \sec x|}{8} + C \]

**Extra credit!** - Please let me know if you can find a shorter solution! -

Solving these kinds of trigonometric integrals can always be done, although as is to be expected with the trigonometric functions, one often finds oneself going around in circles. ;-)

When you’re confronted with an integral of the form \( \int \tan^m x \cdot \sec^n x \, dx \), use the following strategy:

- If \( n \) is even, use the substitution \( u = \tan x \).
- If \( m \) is odd, use the substitution \( u = \sec x \).
- Otherwise, try to find a way to use tabular integration (or integration by parts) to reduce the original integral to one with smaller degrees.

**Integrals of Products of Sines and Cosines** \( \int \sin mx \cos nx \, dx \)

After all that, our final type of integral will seem quite tame. In fact, it isn’t really new either. Tabular integration is the easiest way to solve integrals of the type \( \int \sin(mx) \cos(nx) \, dx \), for positive integer values of \( m \) and \( n \). We’ll solve all of them at once. For less writing, let \( s_m = \sin(mx) \) and \( c_n = \cos(nx) \), so that for \( k \) equal to either \( m \) or \( n \) we have: \( s_k^2 + c_k^2 = 1 \), \( s_k' = kc_k \), \( c_k' = -ks_k \).
Example 16. \[ \int \sin(mx) \cos(nx) \, dx = \int s_m c_n \, dx = ? \] (Assume here that \( m \neq n \).)

\begin{align*}
+ & \quad s_m & c_n \\
- & \quad mc_m & \frac{s_n}{n} \\
+ & \quad -m^2 s_m & -\frac{c_n}{n^2}
\end{align*}

Notice the appearance of our original integral in that last row of the tableaux. Thus we get:

\[
\int s_m c_n \, dx = \frac{s_m s_n}{n} + \frac{mc_m c_n}{n^2} + \frac{m^2}{n^2} \cdot \int s_m c_n \, dx
\]

\[
\left(1 - \frac{m^2}{n^2}\right) \int s_m c_n \, dx = \frac{s_m s_n}{n} + \frac{mc_m c_n}{n^2} \quad \Rightarrow \quad \frac{n^2 - m^2}{n^2} \cdot \int s_m c_n \, dx = \frac{ns_m s_n + mc_m c_n}{n^2}
\]

\[
\int \sin(mx) \cos(nx) \, dx = \frac{n \sin(mx) \sin(nx) + m \cos(mx) \cos(nx)}{n^2 - m^2} + C
\]

That was a bit of a hassle! There is an easier way to do the above problem, based on the trigonometric identity:

\[ \sin(mx) \cos(nx) = \frac{\sin[(m+n)x] + \sin[(m-n)x]}{2} \]

This is usually written without so many parentheses, but evaluated as the above parentheses suggest:

\[ \sin mx \cos nx = \frac{\sin (m+n)x + \sin (m-n)x}{2} \]

We’ll prove that identity, which perhaps you don’t recall (you learned it at your mother’s knee so long ago). We’ll expand the right side, using the more well-known identity,

\[ \sin (a \pm b) = \sin a \cos b \pm \cos a \sin b \].

Then,

\[
\frac{\sin(mx + nx) + \sin(mx - nx)}{2} = \frac{\sin mx \cos nx + \cos mx \sin nx + \sin mx \cos nx - \cos mx \sin nx}{2}
\]

and the identity is proved!

Next, we’ll resolve that last integral using this new trig identity, valid for all integer values of \( m \) and \( n \).
Example 16 (Revisited!). \( \int \sin mx \cos nx \, dx = ? \)

\[
\int \frac{\sin((m+n)x) + \sin((m-n)x)}{2} \, dx = \frac{1}{2} \int \sin((m+n)x) \, dx + \frac{1}{2} \int \sin((m-n)x) \, dx
\]

Assuming \( m \neq n \),

\[
= \frac{1}{2} \cdot \frac{-\cos((m+n)x)}{m+n} + \frac{1}{2} \cdot \frac{-\cos((m-n)x)}{m-n} + C = \frac{-1}{2} \cdot \left( \frac{\cos((m+n)x)}{m+n} + \frac{\cos((m-n)x)}{m-n} \right) + C
\]

or

\[
\left( \frac{m-n}{m+n} \right) \cos((m+n)x) + \left( \frac{m+n}{m-n} \right) \cos((m-n)x)
\]

For the remaining case where \( m = n \),

\[
\int \sin mx \cos mx \, dx = \frac{1}{2} \int \sin(2mx) \, dx = \frac{1}{2} \int \sin(2mx) \, dx + \frac{1}{2} \int \sin(0) \, dx
\]

\[
= \frac{1}{2} \cdot \frac{-\cos(2mx)}{2m} + C
\]

Notice that the definite integral of either of the above cases, from 0 to \( 2\pi \), is always 0, since the sine and cosine functions integrated over a whole number of periods is always 0. For both of these functions spend as much time above the \( x \)-axis as it does below it, and the positive areas are cancelled out by the negative ones. Recall that the period of \( \sin nx \) and of \( \cos nx \) is \( \frac{2\pi}{n} \), so each of these goes through \( n \) complete periods as \( x \) goes from 0 to \( 2\pi \).

Next we’ll consider integrals of the form \( \int \sin mx \sin nx \, dx \). For these it’s easiest if we use the trigonometric identity (verify that it’s an identity!):

\[
\sin mx \sin nx = \frac{-\cos((m+n)x) + \cos((m-n)x)}{2}
\]
Example 17. \[ \int \sin mx \sin nx \, dx = \int \frac{- \cos (m + n)x + \cos (m - n)x}{2} \, dx \]

\[= \frac{1}{2} \left( -\int \cos (m+n)x \, dx + \int \cos (m-n)x \, dx \right) = \frac{1}{2} \int \frac{- \sin (m+n) + \sin (m-n)}{m+n} \, dx + C \]

In the case where \( m = n \),

\[\int \sin mx \sin mx \, dx = \int \frac{- \cos(m+m)x + \cos(m-m)x}{2} \, dx = -\frac{1}{2} \cdot \int \cos 2mx \, dx + \frac{1}{2} \cdot \int \cos 0 \, dx \]

\[= -\frac{1}{2} \cdot \frac{\sin 2mx}{2m} + \frac{1}{2} \cdot \int 1 \, dx = -\frac{\sin 2mx}{4m} + \frac{x}{2} + C \]

\[\int \sin^2 mx \, dx = -\frac{\sin 2mx}{4m} + \frac{x}{2} + C \]

Notice that the definite integral of our first case \((m \neq n)\), from 0 to \(2\pi\), is always 0, since the sine function \(\sin 2mx\) integrated over a whole number of periods is 0. But in the second case, where \(m = n\), the first term is zero when evaluated from 0 to \(2\pi\), but the second term \(\frac{x}{2}\) is not zero when evaluated over a period. This makes sense since our integrand is a square, and so is always positive (when not 0), so the integral is positive over every interval \([A, B]\).

![Graph of \(y = \sin^2 9x\) from 0 to \(2\pi\).](image)

The function \(y = \sin^2 9x\) graphed from 0 to \(2\pi\). The area is positive.

This graph goes through 18 periods of the function.

Lastly we consider integrals of the form \(\int \cos mx \cos nx \, dx\). For these it’s easiest if we use the trig identity (verify that it’s an identity!): \[\cos mx \cos nx = \cos (m + n)x + \cos (m - n)x \]

Example 18. \[\int \cos mx \cos nx \, dx = \int \frac{\cos (m + n)x + \cos (m - n)x}{2} \, dx \]

\[= \frac{1}{2} \left( \int \cos (m+n)x \, dx + \int \cos (m-n)x \, dx \right) = \frac{1}{2} \int \frac{\sin (m+n) + \sin (m-n)}{m+n} \, dx + C \]

In the case where \(m = n\),

\[\int \cos mx \cos mx \, dx = \int \frac{\cos (m+m)x + \cos (m-m)x}{2} \, dx = \frac{1}{2} \cdot \int \cos 2mx \, dx + \frac{1}{2} \cdot \int \cos 0 \, dx \]

\[= \frac{1}{2} \cdot \frac{\sin 2mx}{2m} + \frac{1}{2} \cdot \int 1 \, dx = \frac{\sin 2mx}{4m} + \frac{x}{2} + C \]
\[
\int \cos^2 mx \, dx = \frac{\sin 2mx}{4m} + \frac{x}{2} + C
\]

Notice that the definite integral of our first case \((m \neq n)\), from 0 to \(2\pi\), is always 0, since the sine function \(\sin 2mx\) integrated over a whole number of periods is 0. But in the second case, where \(m = n\), the first term is zero when evaluated from 0 to \(2\pi\), but the second term \(\frac{x}{2}\) is not zero when evaluated over a period. This makes sense since our integrand is a square, and so is always positive (when not 0), so the integral is positive over every interval \([A, B]\).

These last integrals we’ve considered are at the heart of Fourier Analysis, which has very important applications in mathematics, physics, computer science, and engineering (information theory and signal processing in particular).