1. \( \int_{1}^{\infty} \frac{1}{x^e} \, dx = \) ?  
This is a \( p \)-integral with \( p = e \cong 2.718 > 1 \).

Thus the integral from 1 to \( \infty \) converges, and \( \int_{1}^{\infty} \frac{1}{x^e} \, dx = \frac{1}{e-1} \).

(See Example 3)

2. \( \int_{0}^{1} \frac{1}{x^e} \, dx = \) ?  
This is a \( p \)-integral with \( p = e \cong 2.718 > 1 \).

Thus the integral from 0 to 1 diverges, and \( \int_{0}^{1} \frac{1}{x^e} \, dx = \infty \).

3. \( \int_{1}^{\infty} \frac{1}{x^{\pi - e}} \, dx = \) ?  
This is a \( p \)-integral with \( p = \pi - e \cong 0.423310826 < 1 \), less than 1.

Thus the integral diverges, and \( \int_{1}^{\infty} \frac{1}{x^{\pi - e}} \, dx = \infty \).

4. \( \int_{0}^{1} \frac{1}{x^{\pi - e}} \, dx = \) ?  
This is a \( p \)-integral with \( p = \pi - e \cong 0.423310826 < 1 \).

Thus the integral converges, and \( \int_{0}^{1} \frac{1}{x^{\pi - e}} \, dx = -\frac{1}{\pi - e - 1} \).

5. \( \int_{0}^{1} \frac{1}{\sqrt{1-x}} \, dx = \) ?  
Let \( u = 1-x \), \( du = -dx \), and our integral becomes:

\[
\int_{0}^{1} \frac{1}{\sqrt{1-x}} \, dx = -\int_{1}^{0} \frac{1}{\sqrt{u}} \, du = \int_{1}^{0} \frac{1}{u^{\frac{1}{2}}} \, du = \lim_{h \to 0^+} \int_{1}^{1-h} \frac{1}{u^{\frac{1}{2}}} \, du = \lim_{h \to 0^+} \left( 2 \cdot \frac{1}{h^{\frac{1}{2}}} \right) = 2 \cdot \lim_{h \to 0^+} \left( \frac{1}{h^{\frac{1}{2}}} \right) = 2 \cdot \left( \frac{1}{0^{\frac{1}{2}}} \right) = 2
\]
6. \( \int_{0}^{5} \frac{dx}{\sqrt{5 - 2x}} = ? \) Let \( u = 5 - 2x, \ du = -2dx, \) and our integral becomes:

\[
\int_{0}^{5} \frac{1}{\sqrt{5 - 2x}} dx = -\frac{1}{2} \int_{0}^{5} \frac{1}{\sqrt{u}} du = -\frac{1}{2} \lim_{h \to 0^+} \left[ \frac{1}{2} \sqrt{u} \right]_{h}^{5} = \frac{1}{2} \lim_{h \to 0^+} \left( 5^{\frac{1}{2}} - h^{\frac{1}{2}} \right) = \frac{5^{\frac{1}{2}} - 0^{\frac{1}{2}}}{2} = \sqrt{5}
\]

7. \( \int_{0}^{1} \frac{dx}{1 - x} = ? \) Let \( u = 1 - x, \ du = -dx, \) and our integral becomes:

\[
\int_{0}^{1} \frac{1}{1 - x} dx = \lim_{h \to 0^+} \left[ \ln |u| \right]_{h}^{1} = \lim_{h \to 0^+} \left( \ln 1 - \ln h \right) = - \lim_{h \to 0^+} h = -(-\infty) = \infty \quad \text{So the integral diverges.}
\]

8. \( \int_{-\infty}^{0} \frac{1}{x^2 + 1} dx = ? \) This integral converges…

\[
\int_{-\infty}^{0} \frac{1}{x^2 + 1} dx = \lim_{n \to \infty} \int_{-n}^{0} \frac{1}{x^2 + 1} dx = \lim_{n \to \infty} \left[ \tan^{-1} x \right]_{-n}^{0} = \lim_{n \to \infty} \left( \tan^{-1} 0 - \tan^{-1} (-n) \right) = \lim_{n \to \infty} \left( 0 + \tan^{-1} n \right) = \frac{\pi}{2}
\]

The graph of the arctangent function makes the above limit easy to see (literally!).

With that graph in mind, this makes sense:

\[
\int_{-\infty}^{0} \frac{1}{x^2 + 1} dx = \tan^{-1} 0 - \tan^{-1} (-\infty) = 0 - \left( -\frac{\pi}{2} \right) = \frac{\pi}{2}
\]

9. \( \int_{0}^{1} \frac{\ln x}{x} dx = ? \) Since \( \frac{\ln x}{x} \geq \frac{1}{x}, \) and \( \int_{0}^{1} \frac{1}{x} dx \) diverges (it’s the \( p \)-integral with \( p=1 \)), hence \( \int_{0}^{1} \frac{\ln x}{x} dx \) also diverges.
10. \( \int_0^e \ln x \, dx = ? \) Using tabular integration,

\[
\begin{array}{c|c|c}
+ & \ln x & 1 \\
\downarrow & \downarrow & \\
- & \frac{1}{x} & x \\
\end{array}
\]

Our integral becomes:

\[
\int_0^e \ln x \, dx = \lim_{h \to 0^+} \frac{e}{h} \ln x \bigg]_0^h - \lim_{h \to 0^+} \frac{e}{h} \int_0^h \ln x \, dx + \lim_{h \to 0^+} \frac{e}{h} \ln x \bigg]_h^e
\]

\[
= \lim_{h \to 0^+} (e \ln e - h \ln h) - \lim_{h \to 0^+} (e - h) = \lim_{h \to 0^+} (e - h \ln h) - e = \lim_{h \to 0^+} h \ln h
\]

Our next step is a standard trick. By making the variable change \( h = \frac{1}{n} \), we can recast the limit so we can invoke l'Hôpital’s rule. Then:

\[
\lim_{n \to \infty} \frac{h \ln h}{n} = \lim_{n \to \infty} \frac{1}{n} \ln 1 - \ln n = \lim_{n \to \infty} 0 - \ln n = -\lim_{n \to \infty} \ln n
\]

Invoking old l'Hôpital, we get:

\[
\int_0^e \ln x \, dx = -\lim_{n \to \infty} \frac{\ln n}{n} = -\lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{1} = -\lim_{n \to \infty} \frac{1}{n} = -0 = 0
\]

11. \( \int_{\ln(100)}^{\infty} \frac{dx}{x \cdot \ln x \cdot \ln(\ln x)} = ? \) Those logs aren’t enough to push this integral over to convergence…

(See Example 19). Letting \( u = \ln(\ln x) \), \( du = \frac{1}{x \cdot \ln x} \, dx = \frac{dx}{\ln x} \cdot \frac{1}{x} \), our integral becomes:

\[
\int_{\ln(\ln(100))}^{\infty} \frac{du}{u} = \lim_{n \to \infty} \int_{\ln(\ln(100))}^{n} \frac{du}{u} = \lim_{n \to \infty} \ln u \bigg|_{\ln(\ln(100))}^{n} = \lim_{n \to \infty} \ln n - \ln(\ln(\ln(100))) = \infty
\]
12. \( \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx = ? \) Notice that if we make the substitution 
\[ u = \ln x, \quad du = \frac{1}{x} \, dx, \] we’re short an \( x \) in the denominator.

Since \( u = \ln x \implies e^u = x \), we may replace that \( x \) in the denominator by \( e^u \), and our integral becomes:

\[ \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx = \int_{0}^{\infty} \frac{u}{e^u} \, du = \int_{0}^{\infty} u e^{-u} \, du \]

Invoking tabular integration,

\[ \begin{array}{c|c}
+ & u \\
\downarrow & \downarrow \\
- & 1 \\
\downarrow & \downarrow \\
0 & e^{-u} \\
\end{array} \]

Our integral becomes:

\[ \int_{0}^{\infty} u e^{-u} \, du = \lim_{n \to \infty} \left[ \left. u e^{-u} \right|_{0}^{n} - \int_{0}^{n} e^{-u} \, du \right] = \lim_{n \to \infty} \left( -u e^{-u} \right)_{0}^{n} + \lim_{n \to \infty} \left( e^{-u} \right)_{0}^{n} = \lim_{n \to \infty} \left( -n e^{-n} - 0 \right) + \lim_{n \to \infty} e^{-n} = 0 - 0 + 1 = 1 \]

13. \( \int_{0}^{\infty} \cos^2 x \, dx = ? \) Graphing the integrand, it becomes clear that this integral diverges.

For the integral equals the area of in infinite number of “humps”, each with area \( \frac{\pi}{2} \).

14. \( \int_{1}^{\infty} x^{\frac{7}{8}} \, dx = ? \) \( \int_{1}^{\infty} \frac{1}{x^{\frac{7}{8}}} \, dx \), a \( p \)-integral with \( p = \frac{7}{8} < 1 \) \implies diverges.
15. \( \int_0^\infty x^p \, dx = ? \) This integral has a problem, no matter what \( p \) equals. For \( \int_0^\infty x^p \, dx = \frac{1}{A} \int_0^\infty x^p \, dx + \frac{1}{B} \int_0^\infty x^p \, dx \), so

when \( p = 1 \) both \( A \) and \( B \) diverge,

when \( p > 1 \) then \( A \) diverges while \( B \) converges, and

when \( p < 1 \) then \( A \) converges while \( B \) diverges. In any case \( \int_0^\infty x^p \, dx \) thus diverges.

16. \( \int_0^\infty e^{-7x} \, dx = ? \)

\[
= \lim_{n \to \infty} \int_0^n e^{-7x} \, dx = \lim_{n \to \infty} \frac{e^{-7x}}{-7} \Bigg|_0^n = -\frac{1}{7} \lim_{n \to \infty} e^{-7n} = -\frac{1}{7} \lim_{n \to \infty} \frac{1}{e^{7n}} = -\frac{1}{7} (0 - 1) = \frac{1}{7}
\]

17. \( \int_0^\infty e^{-\frac{x}{3}} \, dx = ? \)

\[
= \lim_{n \to \infty} \int_0^n e^{-\frac{x}{3}} \, dx = \lim_{n \to \infty} \frac{1}{-\frac{1}{3}} e^{-\frac{x}{3}} \Bigg|_0^n = -3 \lim_{n \to \infty} e^{-\frac{n}{3}} = -3 \lim_{n \to \infty} \frac{1}{e^{\frac{n}{3}}} = -3 \lim_{n \to \infty} (0 - 1) = 3
\]

For problems # 18 – 26, find a smaller integral that diverges or a larger one that converges.

18. \( \int_1^\infty \frac{dx}{x^6 + 1} = ? \) Since \( \frac{1}{x^6 + 1} \leq \frac{1}{x^6} \), and \( \int_1^\infty \frac{dx}{x^6} = \frac{1}{6 - 1} = \frac{1}{5} \) converges,

it follows that \( \int_1^\infty \frac{dx}{x^6 + 1} \) converges to some value between 0 and \( \frac{1}{5} \).

19. \( \int_0^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx = ? \) We can split this integral up into the sum of a proper integral and an improper one:
\[ \int_0^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx = \frac{1}{2} \int_0^1 \frac{\sqrt{x}}{x^2 + 1} \, dx + \int_1^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx \]

The first integral \( \int_0^1 \frac{\sqrt{x}}{x^2 + 1} \, dx \) is between 0 and 1, since \( 0 \leq \frac{\sqrt{x}}{x^2 + 1} \leq \frac{1}{0+1} = 1 \). As to the second and improper integral, consider:

\[ \frac{\sqrt{x}}{x^2 + 1} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{2} = \frac{1}{2} \]

Hence

\[ \int_1^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx \leq \int_1^\infty \frac{1}{x^2} \, dx = \frac{1}{3} - \frac{1}{2} = \frac{1}{6} = 2. \]

Thus our integral,

\[ \int_0^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx = \int_0^1 \frac{\sqrt{x}}{x^2 + 1} \, dx + \int_1^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx \leq 1 + 2 = 3, \]

converges to a number between 0 and 3.

---

20. \( \int_1^\infty e^{-x} \sin x \, dx \) = ?

First a comparison test.

\[ \int_1^\infty e^{-x} \sin x \, dx \leq \int_1^\infty e^{-x} \cdot 1 \, dx = \int_1^\infty e^{-x} \, dx = \lim_{n \to \infty} \int_1^n e^{-x} \, dx = \lim_{n \to \infty} e^0 - e^{-1} = -\lim_{n \to \infty} \left( \frac{1}{e^n} - \frac{1}{e} \right) = - \left( 0 - \frac{1}{e} \right) = \frac{1}{e} \]

So we know that \( \int_1^\infty e^{-x} \sin x \, dx \) converges to something less than \( \frac{1}{e} \). Let’s solve our integral exactly.

Using tabular integration for \( \int_1^\infty e^{-x} \sin x \, dx = \lim_{n \to \infty} \int_1^n e^{-x} \, dx \), we get:

\[ \begin{array}{ccc}
+ & \sin x & e^{-x} \\
\downarrow & \downarrow & \downarrow \\
- & \cos x & -e^{-x} \\
\downarrow & \downarrow & \downarrow \\
+ & -\sin x & e^{-x} \\
\end{array} \]

So:

\[ \int_1^n e^{-x} \sin x \, dx = \left( -e^{-x} \sin x - e^{-x} \cos x \right) \bigg|_1^n - \int_1^n e^{-x} \sin x \, dx \]
21. \( \int_0^\infty e^{-x} \tan^{-1} x \, dx = \) 

Since \( \tan^{-1} x < \frac{\pi}{2} \), for all \( x \) (see the graph in problem #8’s solution),

\[
\int_0^\infty e^{-x} \tan^{-1} x \, dx \leq \int_0^\infty e^{-x} \frac{\pi}{2} \, dx = \frac{\pi}{2} \cdot \int_0^\infty e^{-x} \, dx = \frac{\pi}{2} \cdot \lim_{n \to \infty} \frac{e^{-x}}{0}
\]

Thus:

\[
\frac{\pi}{2} \cdot \lim_{n \to \infty} (e^{-x} - 1) = -\frac{\pi}{2} \cdot \lim_{n \to \infty} \left( \frac{1}{e^{2n}} - 1 \right) = -\frac{\pi}{2} \cdot (0 - 1) = \frac{\pi}{2}
\]

So our integral also converges, to something less than half of \( \pi \).

22. \( \int_1^\infty (2x^2 + 3x - 1) e^{-x} \, dx = \) 

Notice that for \( 2x^2 \geq 3x - 1 \), for all \( x \geq 1 \). Thus:

\[
I = \int_1^\infty (2x^2 + 3x - 1) e^{-x} \, dx \leq \int_1^\infty (2x^2 + 2x) e^{-x} \, dx = \int_1^\infty 4x^2 e^{-x} \, dx = 4 \cdot \lim_{n \to \infty} \int_1^\infty x^2 e^{-x} \, dx
\]

Tabular integration then yields:

And our integral becomes:
\[ I \leq 4 \cdot \lim_{n \to \infty} \int_1^n x^2 e^{-x} \, dx = 4 \cdot \lim_{n \to \infty} \left( -x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} \right) \bigg|_1^n = -4 \cdot \lim_{n \to \infty} \left( x^2 + 2x + 2 \right) e^{-x} \bigg|_1^n \]

\[ I \leq -4 \cdot \lim_{n \to \infty} \left( \frac{x^2 + 2x + 2}{e^n} \right) \bigg|_1^n = -4 \cdot \lim_{n \to \infty} \left( \frac{n^2 + 2n + 2}{e^n} - \frac{1^2 + 2 \cdot 1 + 2}{e^1} \right) = -4 \cdot \lim_{n \to \infty} \left( \frac{n^2 + 2n + 2}{e^n} - \frac{5}{e} \right) \]

\[ I \leq \frac{20}{e} - 4 \cdot \lim_{n \to \infty} \frac{n^2 - 2n - 2}{e^n} \]

\[ \text{[\text{Hoptial's Rule}]} \]

\[ \downarrow \]

\[ \frac{20}{e} - 4 \cdot \lim_{n \to \infty} \frac{2n^2 - 2}{e^n} = \frac{20}{e} - 4 \cdot \lim_{n \to \infty} \frac{4n}{e^n} = \frac{20}{e} - 0 = \frac{20}{e} \]

So our integral also converges, to something less than \( \frac{20}{e} \).

We could have solved this one exactly by breaking the original integral into 3 integrals, and integrating each of them separately (if you do this you’ll get \( \frac{15}{e} \)). But that wasn’t asked for!

23. \( \int_0^\infty e^{2x} \cdot e^{-x^2} \, dx = ? \)

Since \( e^{2x} \cdot e^{-x^2} \leq x \cdot e^{x^2} \cdot e^{-x^2} \), and \( \int_0^\infty x e^{-x^2} \, dx \) converges (see below), it follows that \( \int_0^\infty e^{2x} \cdot e^{-x^2} \, dx \) converges.

\[ \int_0^\infty x e^{-x^2} \, dx = \frac{1}{2} \int_0^\infty e^u du = -\frac{1}{2} \cdot \lim_{n \to \infty} \int_0^n e^u du = -\frac{1}{2} \cdot \lim_{n \to \infty} \left( e^n \right) \bigg|_0^n = -\frac{1}{2} \cdot \lim_{n \to \infty} (e^n - e^0) = -\frac{1}{2} \cdot \lim_{n \to \infty} \left( \frac{1}{e^n} - 1 \right) = -\frac{1}{2} \cdot (0 - 1) = \frac{1}{2} \]

Notice here that it’s the infinite tail of the integral that determines convergence or divergence here.

\[ \int_0^\infty e^{2x} \cdot e^{-x^2} \, dx = \int_0^4 e^{2x} \cdot e^{-x^2} \, dx + \int_4^\infty e^{2x} \cdot e^{-x^2} \, dx \leq 4 \cdot e + \frac{1}{2} \approx 11.373127 \]
We know that

\[
\int_{0}^{4} e^{2x} \cdot e^{-x^2} \, dx \leq 4e,
\]

since the integrand attains a maximum of \(e\) when \(x = 1\) (Prove this!):

24. \(\int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, dx = ?\)

This splits into two improper integrals:

\[
\int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \int_{0}^{1} \frac{\sin^2 x}{x^2} \, dx + \int_{1}^{\infty} \frac{\sin^2 x}{x^2} \, dx.
\]

We’ll deal with the second of these improper integrals first.

Since \(\frac{\sin x}{x^2} \leq \frac{1}{x^2}\), and

\[
\int_{1}^{\infty} \frac{dx}{x^2} = \frac{1}{2} - 1 = 1
\]

converges, it follows that \(\int_{1}^{\infty} \frac{\sin^2 x}{x^2} \, dx\) converges to something between 0 and 1.

As to the first improper integral, first recall that \(\sin x < x\), because the arclength around the unit circle from the point (1,0) to P is always longer than the y-coordinate of P. Thus \(\sin^2 x < x^2\), and so for all \(x\),

\[
\frac{\sin^2 x}{x^2} < 1.
\]

Also we know that

\[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^2 = \left( \lim_{x \to 0} \frac{\sin x}{x} \right)^2 = 1^2 = 1.
\]

Thus

\[
\int_{0}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \int_{0}^{1} \frac{\sin^2 x}{x^2} \, dx + \int_{1}^{\infty} \frac{\sin^2 x}{x^2} \, dx \leq 1 + 1 = 2,
\]

and our integral converges to a value between 0 and 2.
25. \[ \int_{0}^{\infty} \left( \frac{1}{x^2 + x} \right) \, dx = \int_{0}^{\infty} \left( \frac{x+1}{x(x+1)} - \frac{x}{x(x+1)} \right) \, dx = \int_{0}^{\infty} \frac{x+1-x}{x(x+1)} \, dx = \int_{0}^{\infty} \frac{1}{x^2 + x} \, dx \]

Our integrand, \( \frac{1}{x^2 + x} \leq \frac{1}{x^2} \), and \( \int_{1}^{\infty} \frac{dx}{x^2} = \frac{1}{2-1} = 1 \), hence our integral converges to some value between 0 and 1.

26. \[ \int_{0}^{\infty} \left( \frac{1}{2x-1} - \frac{1}{2x+1} \right) \, dx = ? \]

\[ = \int_{0}^{\infty} \frac{2x+1-(2x-1)}{4x^2-1} \, dx = \int_{0}^{\infty} \frac{2x+1-2x+1}{4x^2-1} \, dx = \frac{2}{4x^2-1} \, dx = \frac{1}{3} \int_{0}^{\infty} \frac{dx}{x^2} = \frac{1}{3} \cdot \frac{1}{2-1} = \frac{1}{3} \]

Since \( 4x^2-1 \geq 3x^2 \), for \( x \geq 1 \), we have our integrand,

\[ \frac{1}{4x^2-1} \leq \frac{1}{3x^2} \], and \( \int_{1}^{\infty} \frac{dx}{x^2} = \frac{1}{3} \cdot \frac{1}{2-1} = \frac{1}{3} \).

Thus: \[ \int_{0}^{\infty} \left( \frac{1}{2x-1} - \frac{1}{2x+1} \right) \, dx = 2 \cdot \int_{0}^{\infty} \frac{dx}{4x^2-1} \leq 2 \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3} \]

And so our integral converges to some value between 0 and \( \frac{2}{3} \).

27. Find the area between the graphs of \( y = \sec x \) and \( y = \tan x \), for \( 0 \leq x \leq \frac{\pi}{2} \).

It often helps to visualize the situation before starting to compute.

From the plot, you can see that both functions have vertical asymptote \( x = \frac{\pi}{2} \).

Our integral is thus improper:
This time, we’ll express the integrand in terms of sines and cosines before attempting the integral. It seems easier this way!

\[
\text{Area} = \int_{0}^{\pi/2} (\sec x - \tan x) \, dx = \lim_{b \to \pi/2} \int_{0}^{b} (\sec x - \tan x) \, dx = \lim_{b \to \pi/2} \int_{0}^{b} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \, dx = \lim_{b \to \pi/2} \int_{0}^{b} \frac{1 - \sin x}{\cos x} \, dx
\]

\[
= \lim_{b \to \pi/2} \int_{0}^{b} \frac{1 - \sin x}{\cos x} \cdot \frac{(1 + \sin x)}{(1 + \sin x)} \, dx = \lim_{b \to \pi/2} \int_{0}^{b} \frac{1 - \sin^2 x}{\cos x (1 + \sin x)} \, dx = \lim_{b \to \pi/2} \int_{0}^{b} \frac{\cos^2 x}{\cos x (1 + \sin x)} \, dx
\]

\[
\text{Area} = \lim_{b \to \pi/2} \int_{0}^{b} \frac{\cos x}{1 + \sin x} \, dx
\]

Now we make the substitution \( u = 1 + \sin x \), \( du = \cos x \, dx \), and our integral becomes:

\[
\text{Area} = \lim_{b \to \pi/2} \int_{0}^{b} \frac{\cos x}{1 + \sin x} \, dx = \lim_{b \to \pi/2} \int_{1}^{1 + \sin b} \frac{du}{u} = \lim_{b \to \pi/2} \ln u \bigg|_{1}^{1 + \sin b} = \lim_{b \to \pi/2} \left( \ln (1 + \sin b) - \ln 1 \right)
\]

\[
= \lim_{b \to \pi/2} \ln (1 + \sin b) = \ln \left( 1 + \sin \frac{\pi}{2} \right) \ln (1 + 1) = \ln 2
\]

\[
\text{Area} = \ln 2
\]

28. Find the area between the graphs of \( y = \frac{x + 1}{1 - x^2} \) and \( y = \frac{x + 1}{2 - x - x^2} \), for \( 0 \leq x \leq 1 \).

It often helps to visualize the situation before starting to compute. From the plot, you can see that both functions have a vertical asymptote at \( x = 1 \). Our area integral is thus improper:

\[
\int_{0}^{1} \left( \frac{x + 1}{1 - x^2} - \frac{x + 1}{2 - x - x^2} \right) \, dx = \lim_{n \to 1} \int_{0}^{n} \left( \frac{x + 1}{1 - x^2} - \frac{x + 1}{2 - x - x^2} \right) \, dx
\]

\[
= \lim_{n \to 1} \left( \int_{0}^{n} \frac{x}{1 - x^2} \, dx - \int_{0}^{n} \frac{x + 1}{2 - x - x^2} \, dx \right)
\]

\[
= \lim_{n \to 1} \left( \int_{0}^{n} \frac{x}{1 - x^2} \, dx + \int_{0}^{n} \frac{dx}{1 - x^2} \right)
\]

\[
= \text{Integral 1} + \text{Integral 2}
\]

\[
\int_{0}^{n} \frac{x + 1}{2 - x - x^2} \, dx
\]

\[
= \text{Integral 3}
\]
Integral 1: let $u = 1 - x^2$, $du = -2xdx$, $\Rightarrow -\frac{1}{2} du = xdx$. Then,

\[
\int_0^n \frac{x}{1-x^2} \, dx = -\frac{1}{2} \int_0^n \frac{du}{u} = -\frac{1}{2} \ln|u| \bigg|_0^n = -\frac{1}{2} \left( \ln(1-n^2) - \ln 1 \right) = -\frac{1}{2} \ln(1-n^2).
\]

Integral 2: $\int_0^n \frac{dx}{1-x^2} \, dx = \tan^{-1} x \bigg|_0^n = \tan^{-1} n - \tan^{-1} 0 = \tan^{-1} n$.

Integral 3: For this one we’ll use partial fractions:

\[
\frac{x+1}{2-x-x^2} = \frac{x+1}{(2+x)(1-x)} = \frac{A}{2+x} + \frac{B}{1-x}
\]

Clearing denominators, we get:

\[
x + 1 = A(1 - x) + B(2 + x)
\]

$x = -2 \rightarrow -1 = A(1 + 2) + B \cdot 0 \Rightarrow -1 = 3A \Rightarrow A = -\frac{1}{3}$

$x = 1 \rightarrow 1 = A \cdot 0 + B(2 + 1) \Rightarrow 1 = 0 + 3B \Rightarrow B = \frac{1}{3}$

\[
\Rightarrow \frac{x+1}{2-x-x^2} = \frac{-\frac{1}{3}}{2+x} + \frac{\frac{1}{3}}{1-x} = \frac{1}{3} \left( \frac{1}{1-x} - \frac{1}{2+x} \right)
\]

And so,

\[
\int_0^n \frac{x+1}{2-x-x^2} \, dx = \frac{1}{3} \left( \int_0^n \frac{dx}{1-x} - \int_0^n \frac{dx}{2+x} \right) = \frac{1}{3} \left( \ln(1-x) - \ln(2+x) \right) \bigg|_0^n
\]

\[
= \frac{1}{3} \left( \ln(1-n) - \ln(2+n) - \ln(1) + \ln(2) \right)
\]

\[
= \frac{1}{3} \left( \ln(1-n) - \ln(2+n) + \ln 2 \right) = \frac{1}{3} \ln \left( \frac{1-n}{2+n} \right) + \ln 2 \frac{3}{3}
\]

Thus:

\[
\int_0^n \left( \frac{x+1}{1-x^2} - \frac{x+1}{2-x-x^2} \right) \, dx = \lim_{n \to 1} \left( \int_0^n \frac{x}{1-x^2} \, dx + \int_0^n \frac{dx}{1-x^2} - \int_0^n \frac{x+1}{2-x-x^2} \, dx \right)
\]

\[
= \lim_{n \to 1} \left( -\frac{\ln(1-n^2)}{2} + \tan^{-1} n - \left( \frac{1}{3} \ln \left( \frac{1-n}{2+n} \right) + \ln 2 \frac{3}{3} \right) \right)
\]

\[
= \lim_{n \to 1} \left( -\frac{1}{2} \ln(1-n^2) + \tan^{-1} n - \frac{1}{3} \ln \left( \frac{1-n}{2+n} \right) - \ln 2 \frac{3}{3} \right)
\]

\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]

\[
= \infty + \frac{\pi}{4} - \infty - \frac{\ln 2}{3} = \infty
\]