8.3 Area of a Surface of Revolution

We’ll start by reviewing volumes of revolution, and then proceed to finding the areas swept out by curves revolved about a fixed line. The key to understanding these topics is in the geometry. One should always first sketch the curve, and then visualize the volume that results when the curve is revolved about a line. The calculus should flow right out of your diagram, if you draw it correctly. Before we begin, there are some formulas you’ll need to recall from elementary geometry.

The above are the basis for the disk, washer, and shell methods you learned in Calculus I to find volumes of surfaces of revolution. We’ll next quickly review each of these methods.

**Disk Method**

The volume of the circular disk at \( x \), with radius \( f(x) \) and thin width \( dx \), is given by:

\[
V_s = \pi f(x)^2 \, dx
\]
Adding up all the thin (with width $dx$) disk volumes from $a$ to $b$, we get the volume of revolution obtained when the region between the $x$-axis and the graph of $f$ between $x = a$ and $x = b$ is revolved about the $x$-axis:

$$V = \pi \int_a^b f(x)^2 \, dx$$

**Washer Method**

![Washer Method Diagram]

The volume of the circular washer at $x$, with radii $f(x)$ and $g(x)$, and width $dx$, is given by:

$$V_x = \pi \left( f(x)^2 - g(x)^2 \right) dx$$

Adding up all the thin (with width $dx$) washer volumes for $x$ from $a$ to $b$, we get the volume of revolution obtained when the region bounded by the graphs of $f$ and $g$ between $x = a$ and $x = b$ is revolved about the $x$-axis:

$$V = \pi \int_a^b \left( f(x)^2 - g(x)^2 \right) dx$$

**Shell Method**

![Shell Method Diagram]

The volume of a circular cylindrical “shell” with radius $r = x$, height $h = f(x) - g(x)$, and thickness $w = dx$, is given by:

$$V_s = 2\pi x \left( f(x) - g(x) \right) dx.$$ 

Adding up all the thin (with width $dx$) shell volumes for $x$ from $a$ to $b$, we get the volume of revolution obtained when the region bounded by the graphs of $f$ and $g$ between $x = a$ and $x = b$ is revolved about the $y$-axis:

$$V = 2\pi \int_a^b x \left( f(x) - g(x) \right) dx$$

For further review on volumes of revolution, please see this Calculus I webpage.
There you’ll find examples involving regions being revolved about lines other than the coordinate axes.
Surfaces of Revolution

We’ve been finding the volumes of three-dimensional regions obtained by revolving two-dimensional regions about a line. The two-dimensional region sweeps out a three-dimensional volume. Now we’ll revolve a piece of curve about a line to generate a two-dimensional surface, imbedded in three-dimensional space. The resulting piece of surface has an area which you’ll now see how to express in terms of integrals (and we’ll even solve a couple of the resulting integrals!). We start with a piece of a function’s graph over an interval \( a \leq x \leq b \), and seek the area swept out by that curve as it’s rotated about the \( x \)-axis. Imagine an infinitesimal piece of the curve, of length \( ds = \sqrt{dx^2 + dy^2} \), above \( x \) on the \( x \)-axis, as shown in the figure on the left:

As the curve is swept about the \( x \)-axis, the little piece of curve of length \( ds \) is swept around a circle of radius \( f(x) \) through a circumference of \( 2\pi f(x) \). The area of the resulting circular strip is thus approximately \( 2\pi f(x) ds \) (assuming the radius of the sweep \( f(x) \) is much greater than \( ds \)), and adding up all the strips between \( x=a \) and \( x=b \) (i.e., integrating), we obtain the total area of revolution:

\[
\text{Area} = 2\pi \int_{x=a}^{x=b} f(x) \, ds.
\]

Furthermore,

\[
ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + dy^2} \cdot \frac{dx}{dx} = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \sqrt{1 + \left(f'(x)\right)^2} \cdot dx,
\]

hence our area of revolution integral becomes:

\[
\text{\textbackslash Area} = 2\pi \int_{x=a}^{x=b} f(x) \, ds = 2\pi \int_{a}^{b} f(x) \sqrt{1 + \left(f'(x)\right)^2} \, dx
\]

\[
\text{\textbackslash Area} = 2\pi \int_{a}^{b} f(x) \sqrt{1 + \left(f'(x)\right)^2} \, dx
\]
Example 1. Find the area swept out when the graph of \( f(x) = \sqrt{x} \) between \( x = 1 \) and \( x = 4 \) is revolved about the \( x \)-axis.

Although we could just use the above formula we’ve just derived, I recommend sketching a graph of \( y = \sqrt{x} \) first, and visualizing the shape obtained when the curve is revolved. Also it’s good to show the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x \) is swept around) equals \( \sqrt{x} \), and so the area of the strip is

\[
\text{Area}_{\text{strip}} \approx 2\pi r \cdot ds = 2\pi \sqrt{x} \cdot ds.
\]

(Note: This is the exact area when the little piece of curve \( ds \) is parallel to the \( x \)-axis. But \( ds \) is going to shrink to 0 length, in the area integral, so the error in this approximation shrinks to zero and the resulting integral is valid.)

Recall that \( ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \sqrt{1 + (f'(x))^2} \, dx \). Thus:

\[
f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \Rightarrow \quad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \quad \rightarrow \quad ds = \sqrt{1 + \left( f'(x) \right)^2} \, dx = \sqrt{1 + \left( \frac{1}{2\sqrt{x}} \right)^2} \, dx = \sqrt{1 + \frac{1}{4x}} \, dx
\]

\[
ds = \sqrt{1 + \left( f'(x) \right)^2} \cdot dx = \sqrt{\frac{4x + 1}{4x}} \cdot dx
\]

And so:

\[
\text{Area} = 2\pi \int_{a}^{b} r \, ds = 2\pi \int_{a}^{b} f(x) \sqrt{1 + \left( f'(x) \right)^2} \, dx
\]

\[
= 2\pi \int_{1}^{4} \sqrt{x} \sqrt{\frac{4x + 1}{4x}} \, dx = 2\pi \int_{1}^{4} \frac{4x + 1}{4x} \, dx = 2\pi \int_{1}^{4} \frac{4x + 1}{2} \, dx
\]

\[
\text{Area} = \pi \int_{1}^{4} \sqrt{4x + 1} \, dx
\]

Letting \( u = 4x + 1 \), \( du = 4dx \), our integral becomes:

\[
\text{Area} = \pi \cdot \frac{1}{4} \int_{5}^{17} u \, du = \pi \cdot \frac{1}{4} \left[ u^\frac{3}{2} \right]_{5}^{17} = \pi \cdot \frac{1}{4} \cdot \left( \frac{3}{2} \right) \left( 17^\frac{3}{2} - 5^\frac{3}{2} \right) = \pi \cdot \frac{2}{3} \cdot \left( \frac{3}{2} \cdot 5 \right) = \pi \cdot \frac{5}{2} \left( \sqrt{17^3} - \sqrt{5^3} \right)
\]
Example 2. Find the area swept out when the graph of \( y = x^3 \) between \( x = 0 \) and \( x = \frac{1}{\sqrt{3}} \) is revolved about the \( x \)-axis.

We first sketch a graph of \( y = x^3 \), to help us visualize the shape obtained when the curve is revolved. Notice the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x \) is swept around) equals \( x^3 \), and so the area of the strip is

\[
\text{Area}_{\text{strip}} \approx 2\pi r \cdot ds = 2\pi x^3 \cdot ds .
\]

Recall that \( ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \) \( dx = \sqrt{1 + (f'(x))^2} \) \( dx \). Then:

\[
f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2 \quad \rightarrow \quad ds = \sqrt{1 + (f'(x))^2} \) \( dx = \sqrt{1 + (3x^2)^2} \) \( dx = \sqrt{1 + 9x^4} \) \( dx \)

And so:

\[
\text{Area} = 2\pi \int_a^b r \) \( ds = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \) \( dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} x^3 \sqrt{1 + 9x^4} \) \( dx 
\]

Letting \( u = 1 + 9x^4 \), \( du = 36x^3 \) \( dx \), our integral becomes:

\[
= 2\pi \int_0^{\frac{1}{\sqrt{3}}} x^3 \sqrt{1 + 9x^4} \) \( dx = 2\pi \cdot \frac{1}{36} \int_1^{\frac{1}{\sqrt{3}}} u^2 \) \( du = \frac{\pi}{18} \cdot \frac{1}{3} \int_1^{\frac{1}{\sqrt{3}}} u^2 \) \( du = \frac{\pi}{18} \cdot \frac{3}{3} \left( \frac{3}{2} - 1 \right) = \frac{\pi (2\sqrt{2} - 1)}{27}
\]
Example 3. Find the area swept out when the graph of \( y = e^x \)

between \( x = -1 \) and \( x = 0 \) is revolved about the \( x \)-axis.

To the left see a sketch of \( y = e^x \).

Notice the “strip” whose area is to become our integrand.

The radius of the circular strip (which the little piece of curve \( ds \) above \( x \) is swept around) equals \( e^x \), and so

the area of the strip is \( \text{Area}_{\text{strip}} \approx 2\pi r \cdot ds = 2\pi e^x \cdot ds \).

Recall that \( ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + (f'(x))^2} \, dx \). Thus:

\[
f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x \quad \rightarrow \quad ds = \sqrt{1 + (f'(x))^2} \, dx = \sqrt{1 + (e^x)^2} \, dx = \sqrt{1 + e^{2x}} \, dx
\]

And so, summing up all these area strips from \( x = -1 \) up to \( x = 0 \), we obtain our area integral:

\[
\text{Area} = 2\pi \int_a^b r \, ds = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx = 2\pi \int_{-1}^0 e^x \sqrt{1 + e^{2x}} \, dx
\]

Letting \( u = e^x \), \( du = e^x \, dx \), our integral becomes:

\[
\text{Area} = 2\pi \int_{-1}^0 e^x \sqrt{1 + e^{2x}} \, dx = 2\pi \int_{e^{-1}}^1 \sqrt{1 + u^2} \, du
\]

Next let \( u = \tan \theta \), \( du = \sec^2 \theta \, d\theta \), so our bounds of integration change as follows:

\[
u = e^{-1} \quad \Rightarrow \quad \theta = \tan^{-1}(e^{-1}), \quad \text{and} \quad u = 1 \quad \Rightarrow \quad \theta = \tan^{-1}1 = \frac{\pi}{4}
\]

And our integral becomes:

\[
\text{Area} = 2\pi \int_{e^{-1}}^1 \sqrt{1 + u^2} \, du = 2\pi \int_{\tan^{-1}(e^{-1})}^{\frac{\pi}{4}} \sqrt{1 + \tan^2 \theta \cdot \sec^2 \theta} \, d\theta
\]

\[
= 2\pi \int_{\tan^{-1}(e^{-1})}^{\frac{\pi}{4}} \sec \theta \cdot \sec^2 \theta \, d\theta = 2\pi \int_{\tan^{-1}(e^{-1})}^{\frac{\pi}{4}} \sec^3 \theta \, d\theta
\]
Invoking tabular integration to integrate the secant cubed:

\[
\begin{array}{c|c|c}
+ & \sec x & \sec^2 x \\
\downarrow & \downarrow & \downarrow \\
- & \sec x \tan x & \tan x \\
\end{array}
\]

We get:

\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx
\]

\[
= \sec x \tan x - \int \sec x \left( \sec^2 x - 1 \right) \, dx
\]

\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx
\]

\[
2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx = \sec x \tan x + \ln |\sec x + \tan x|
\]

\[
\Rightarrow \int \sec^3 x \, dx = \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C
\]

Thus, Area \( A = 2\pi \int_{\tan^{-1}(e^{-1})}^{\pi/4} \sec^3 \theta \, d\theta = \frac{\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|}{2} \bigg|_{\theta = \tan^{-1}(e^{-1})}^{\theta = \pi/4} \)

\[
= \pi \cdot \left( \sec \frac{\pi}{4} \cdot \frac{\pi}{4} + \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \sec \left( \tan^{-1}(e^{-1}) \right) \tan \left( \tan^{-1}(e^{-1}) \right) \right. \\
\left. - \ln \left| \sec \left( \tan^{-1}(e^{-1}) \right) + \tan \left( \tan^{-1}(e^{-1}) \right) \right| \right)
\]

\[
= \pi \cdot \left( \sqrt{2} + 1 - \sec \left( \tan^{-1}(e^{-1}) \right) \cdot e^{-1} - \ln \left| \sec \left( \tan^{-1}(e^{-1}) \right) + e^{-1} \right| \right)
\]

\[
= \pi \cdot \left( \sqrt{2} + \ln \left( 1 + \sqrt{2} \right) - \frac{\sec \left( \tan^{-1}(e^{-1}) \right)}{e} - \ln \left( \sec \left( \tan^{-1}(e^{-1}) \right) + \frac{1}{e} \right) \right)
\]

To find \( \sec \left( \tan^{-1}(e^{-1}) \right) = \sec \varphi \), we construct a right

\[
\begin{array}{c}
\varphi \\
\sqrt{1+e^2} \\
1 \\
e \\
\end{array}
\]

triangle with the property that \( \tan \varphi = e^{-1} = \frac{1}{e} \):

And so \( \sec \left( \tan^{-1}(e^{-1}) \right) = \sec \varphi = \frac{\sqrt{1+e^2}}{e} \), and our area result becomes:
Example 4. Find the area swept out when the graph of the hyperbolic cosine between $x = -\ln 2$ and $x = \ln 2$ is revolved about the $x$-axis.

We sketch a graph of $y = \cosh x$ to visualize the shape obtained when the curve is revolved. Notice the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve $ds$ above $x$ is swept around) equals $\cosh x$, and so the area of the strip is

$$Area_{\text{strip}} \approx 2\pi r \cdot ds = 2\pi \cosh x \cdot ds.$$

Thus,

$$f(x) = \cosh x \quad \Rightarrow \quad f'(x) = \sinh x \quad \rightarrow \quad ds = \sqrt{1 + (f'(x))^2} \, dx = \sqrt{1 + \sinh^2 x} \, dx.$$

Notice in that last step that we used the hyperbolic identity $\cosh^2 x = \frac{1 + \cosh 2x}{2}$. And yes, it has the exact same form as the corresponding half-angle identity from trigonometry that we use to integrate the squared cosine! ☺ Continuing to solve our area integral, we get:

$$Area = \pi \cdot \left( \sqrt{2} + \ln (1 + \sqrt{2}) - \frac{\sqrt{1 + e^2}}{e} \cdot \ln \left( \frac{\sqrt{1 + e^2}}{e} + 1 \right) \right)$$

$$= \pi \cdot \left( \sqrt{2} + \ln (1 + \sqrt{2}) - \frac{\sqrt{1 + e^2}}{e^2} - \ln \left( 1 + \sqrt{1 + e^2} \right) + \ln e \right)$$

$$= \pi \cdot \left( 1 + \sqrt{2} + \ln (1 + \sqrt{2}) - \frac{\sqrt{1 + e^2}}{e^2} - \ln \left( 1 + \sqrt{1 + e^2} \right) + 1 \right)$$

$$= \pi \cdot \left( 1 + \sqrt{2} + \left( 1 + \sqrt{2} + 1 \right) - \frac{\sqrt{1 + e^2}}{e^2} - \ln \left( 1 + \sqrt{1 + e^2} \right) \right)$$

$$= \pi \cdot \left( 1 + \sqrt{2} + \ln (1 + \sqrt{2}) - \frac{\sqrt{1 + e^2}}{e^2} - \ln \left( 1 + \sqrt{1 + e^2} \right) \right)$$
\[ \text{Area} = 2\pi \int_{-\ln 2}^{\ln 2} \frac{1 + \cosh 2x}{2} \, dx = \pi \int_{-\ln 2}^{\ln 2} \cosh 2x \, dx = \pi \left( \ln 2 - (-\ln 2) \right) + \frac{\pi}{2} \cdot \sinh 2x \]

Since the sinh is an odd function

\[ = 2\pi \ln 2 + \frac{\pi}{2} \left( \sinh (2 \ln 2) - \sinh (-2 \ln 2) \right) = 2\pi \ln 2 + \frac{\pi}{2} \left( \sinh (2 \ln 2) + \sinh (2 \ln 2) \right) \]

\[ = 2\pi \ln 2 + \frac{\pi}{2} \cdot 2 \sinh (2 \ln 2) = 2\pi \ln 2 + \pi \sinh (2 \ln 2) \]

\[ \int \int \int \]

\[ \text{Areas of Surfaces of Revolution of Parametric Curves} \]

We start with a piece of curve defined by the parametric equations \( \begin{cases} x = x(t) \\ y = y(t) \end{cases} \) over an interval \( a \leq t \leq b \), and seek the area swept out by that curve as it’s rotated about the \( x \)-axis. Imagine an infinitesimal piece of the curve, of length \( ds = \sqrt{dx^2 + dy^2} \), at a point \( t \) in the interval \( a \leq t \leq b \), as shown in the figure on the left:

As the curve is swept about the \( x \)-axis, the little piece of curve of length \( ds \) is swept around a circle of radius \( y(t) \) through a circumference of \( 2\pi \cdot y(t) \). The area of the resulting circular strip is thus approximately \( 2\pi r = 2\pi \cdot y(t) \cdot ds \) (assuming the radius of the sweep \( y(t) \) is much greater than \( ds \)), and adding up all the strips between \( t=a \) and \( t=b \) (i.e., integrating), we obtain the total area of
revolution: \[ Area = 2\pi \int_{a}^{b} r \, ds = 2\pi \int_{a}^{b} y(t) \, ds \] Furthermore,

\[ ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + dy^2} \cdot \frac{dt}{dt} = \sqrt{(dx^2 + dy^2) \left( \frac{1}{dt} \right)^2} \cdot dt = \sqrt{(dx^2 + dy^2)} \cdot dt \]

\[ ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \cdot dt = \sqrt{(x'(t))^2 + (y'(x))^2} \cdot dt \],

hence our area of revolution integral becomes:

\[ \text{Area} = 2\pi \int_{a}^{b} y(t) \, ds = 2\pi \int_{a}^{b} y(t) \sqrt{(x'(t))^2 + (y'(x))^2} \, dt \]

\[ \text{Area} = 2\pi \int_{a}^{b} y(t) \sqrt{(x'(t))^2 + (y'(x))^2} \, dt \]

Example 5. Find the area swept out when the graph of the hypocycloid defined by parametric equations \[
\begin{align*}
    x &= 5\sin^3 t \\
    y &= 5\cos^3 t
\end{align*}
\] over the interval \(0 \leq t \leq \frac{\pi}{2}\) is revolved about the \(x\)-axis.

Note the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x(t) \) is swept around) is equal to \( r = y(t) = 5\cos^3 t \), and so the area of the strip is

\[ \text{Area}_{strip} \approx 2\pi r \, ds = 2\pi \cdot 5\cos^3 t \, ds = 10\pi \cos^3 t \, ds \]

Recall that \( ds = \sqrt{dx^2 + dy^2} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \). Then:

\[
\begin{align*}
    \begin{cases} 
        x = 5\sin^3 t \\
        y = 5\cos^3 t
    \end{cases}
    \Rightarrow 
    \begin{cases} 
        \frac{dx}{dt} = 15\sin^2 t \cdot \cos t \\
        \frac{dy}{dt} = 15\cos^2 t \cdot (-\sin t) = -15\cos^2 t \cdot \sin t
    \end{cases}
\end{align*}
\]
And so:

\[ ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \sqrt{(-15\cos^2 t \cdot \sin t)^2 + (15\sin^2 t \cdot \cos t)^2} \, dt \]

\[ = \sqrt{15^2 \cos^2 t \sin^2 t \left(\cos^2 t + \sin^2 t\right)} \, dt \quad \Rightarrow \quad ds = 15 \cos t \sin t \, dt \]

And so, summing up all these area strips from \( t = 0 \) up to \( t = \frac{\pi}{2} \), we obtain our area integral:

\[ Area = 2\pi \int_{\pi/2}^{\pi} r \cdot ds = 2\pi \int_{0}^{\pi} 5 \cos^3 t \cdot 15 \cos t \sin t \, dt = 150\pi \int_{0}^{\pi} \cos^4 t \sin t \, dt \]

Substituting \( u = \cos x, \ du = -\sin x \, dx \), our integral becomes:

\[ Area = 150\pi \int_{0}^{\pi} \cos^4 t \sin t \, dt = -150\pi \int_{0}^{1} u^4 \, du = 150\pi \left[ u^5 \right]_{0}^{1} = 30 \pi \left( 1^5 - 0^5 \right) \]

\[ Area = 30\pi \]

Example 6. Find the area swept out when the graph of the spiral defined by parametric equations \( \begin{cases} x = e^t \cos \pi t \\ y = e^t \sin \pi t \end{cases} \) over the interval \( 0 \leq t \leq 1 \) is revolved about the \( x \)-axis.

Note the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x(t) \) is swept around) is equal to \( y(t) = e^t \sin \pi t \), and so the area of the strip is

\[ Area_{\text{strip}} \approx 2\pi r \, ds = 2\pi \, e^t \sin \pi t \, ds . \]

Recall that \( ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \). Then:
\[
\begin{aligned}
\begin{cases}
x = e^t \cos \pi t \\
y = e^t \sin \pi t
\end{cases}
\Rightarrow \\
\begin{aligned}
\frac{dx}{dt} &= e^t \cos \pi t - \pi e^t \sin \pi t = e^t \left( \cos \pi t - \pi \sin \pi t \right) \\
\frac{dy}{dt} &= e^t \sin \pi t + \pi e^t \cos \pi t = e^t \left( \sin \pi t + \pi \cos \pi t \right)
\end{aligned}
\end{aligned}
\]

And so:

\[
ds = \sqrt{\left( e^t \left( \cos \pi t - \pi \sin \pi t \right) \right)^2 + \left( e^t \left( \sin \pi t + \pi \cos \pi t \right) \right)^2} \, dt
\]

It really helps to substitute \( c = \cos \pi t \) and \( s = \sin \pi t \), as this protects you from writer’s cramp!

\[
ds = \sqrt{\left( e^t \left( c - \pi s \right) \right)^2 + \left( e^t \left( s + \pi c \right) \right)^2} \, dt = \sqrt{e^{2t} \left( (c - \pi s)^2 + (s + \pi c)^2 \right)} \, dt
\]

\[
= e^t \sqrt{c^2 - 2\pi cs + \pi^2 s^2 + s^2 + 2\pi cs + \pi^2 c^2} \, dt = e^t \sqrt{\frac{c^2 + s^2}{\pi^2} + \pi^2 \left( \frac{c^2 + s^2}{\pi^2} \right)} \, dt
\]

\[
ds = e^t \sqrt{1 + \pi^2} \, dt
\]

And so, summing up all these area strips from \( t = 0 \) up to \( t = 1 \), we obtain our area integral:

\[
\text{Area} = 2\pi \int_{t=0}^{t=1} r \cdot ds = 2\pi \int_{t=0}^{t=1} e^t \sin \pi t \cdot e^t \sqrt{1 + \pi^2} \, dt = 2\pi \sqrt{1 + \pi^2} \cdot \left[ e^t \sin \pi t \right]_{t=0}^{t=1}
\]

Invoking tabular integration to integrate \( I = \int e^t \sin \pi t \, dt \):

\[
\begin{array}{c|c|c}
\text{+} & \sin \pi t & e^{2t} \\
\downarrow & & \downarrow \\
\text{-} & \pi \cos \pi t & e^{2t} \\
\downarrow & & \downarrow \\
\text{+} & -\pi^2 \sin \pi t & e^{2t} \\
\end{array}
\]

That last row contains the original integral, \( I \):

\[
I = \frac{e^{2t} \sin \pi t}{2} - \frac{\pi e^{2t} \cos \pi t}{4} - \frac{\pi^2}{4} \cdot I \quad \Rightarrow \\
I \left( 1 + \frac{\pi^2}{4} \right) = \frac{e^{2t} \sin \pi t}{2} - \frac{\pi e^{2t} \cos \pi t}{4} \\
\Rightarrow I \cdot \frac{4 + \pi^2}{4} = \frac{e^{2t} \sin \pi t}{2} - \frac{\pi e^{2t} \cos \pi t}{4} \quad \Rightarrow \\
I = \frac{2e^{2t} \sin \pi t - \pi e^{2t} \cos \pi t}{4 + \pi^2} = \frac{e^{2t} \left( 2 \sin \pi t - \pi \cos \pi t \right)}{4 + \pi^2}
\]
Thus,

\[
\text{Area} = 2\pi\sqrt{1 + \pi^2} \cdot \left| \int_0^1 e^{2t} \sin \pi t \, dt \right| = 2\pi\sqrt{1 + \pi^2} \cdot \frac{e^{2t}(2 \sin \pi t - \pi \cos \pi t)}{4 + \pi^2} \bigg|_0^1
\]

\[
= \frac{2\pi\sqrt{1 + \pi^2}}{4 + \pi^2} \cdot \left( e^2 (2 \sin \pi - \pi \cos \pi) - e^0 (2 \sin 0 - \pi \cos 0) \right)
\]

\[
= \frac{2\pi\sqrt{1 + \pi^2}}{4 + \pi^2} \cdot \left( e^2 (0 + \pi) - e^0 (0 - \pi) \right) = \frac{2\pi\sqrt{1 + \pi^2}}{4 + \pi^2} \cdot (\pi e^2 + \pi)
\]

\[
\text{Area} \approx \frac{2\pi^2 (e^2 + 1) \sqrt{\pi^2 + 1}}{4 + \pi^2}
\]

When Integrals Can’t be Solved

Almost always when one attempts to calculate a surface area using the techniques of this section one ends up with an integral that can’t be solved. That is to say, we can’t solve the integral in terms of functions for which we have names. In such cases one can resort to numerical integration to get a good approximation to the exact value of the integral. Excellent numerical integration software is readily available, and is hard-wired into many calculators. In the following examples, we’ll just set up the integral that equals the desired area of revolution, but we won’t try to solve the integrals. Numerical answers are also given, for those of you equipped with the appropriate technology!

Example 7. Find an integral that equals the area swept out when the graph of \( y = \frac{1}{x} \) between \( x = 1 \) and \( x = 3 \) is revolved about the \( x \)-axis.

First sketch a graph of \( y = \frac{1}{x} \), to help us visualize the shape obtained when the curve is revolved. Notice the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x \) is swept around) equals \( x^{-1} \), and so the area of the strip is

\[
\text{Area}_{strip} \approx 2\pi r \cdot ds = 2\pi x^{-1} ds.
\]
Recall that 

\[ ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \quad dx = \sqrt{1 + \left( f'(x) \right)^2} \quad dx . \] 

Thus:

\[ f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2} \quad \Rightarrow \quad ds = \sqrt{1 + \left( f'(x) \right)^2} = \sqrt{1 + (-x^{-2})^2} = \sqrt{1 + x^{-4}} \]

And so:

\[ Area = 2\pi \int_a^b r \, ds = 2\pi \int_a^b f(x) \sqrt{1 + \left( f'(x) \right)^2} \, dx = 2\pi \int_1^3 x^{-1} \cdot \sqrt{1 + x^{-4}} \, dx \]

This integral can also be written without negative exponents:

\[ Area = 2\pi \int_1^3 x^{-1} \cdot \sqrt{1 + x^{-4}} \, dx = 2\pi \int_1^3 \frac{1}{x} \cdot \sqrt{1 + \frac{x^4}{x^4}} \, dx = 2\pi \int \frac{1}{x} \cdot \sqrt{\frac{x^4+1}{x^2}} \, dx \]

Area \( \approx 7.603062807809119212 \)

Example 8. Find an integral that equals the area swept out when the graph of \( y = \sqrt[3]{x} \) between \( x = 1 \) and \( x = 27 \) is revolved about the \( x \)-axis.

First sketch a graph of \( y = \sqrt[3]{x} \), to help us visualize the shape obtained when the curve is revolved. Notice the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x \) is swept around) equals \( \frac{1}{x^\frac{2}{3}} \), and so the area of the strip is

\[ Area_{strip} \approx 2\pi r \cdot ds = 2\pi x^{\frac{1}{3}} ds . \]

Recall that 

\[ ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \quad dx = \sqrt{1 + \left( f'(x) \right)^2} \quad dx . \] 

Thus:

\[ f(x) = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3} x^{-\frac{2}{3}} \quad \Rightarrow \quad ds = \sqrt{1 + \left( f'(x) \right)^2} \, dx = \sqrt{1 + \left( \frac{1}{3} x^{-\frac{2}{3}} \right)^2} \, dx = \sqrt{1 + \frac{1}{9x^\frac{4}{3}}} \, dx \]

\[ \Rightarrow \quad ds = \sqrt{\frac{9x^\frac{4}{3} + 1}{9x^\frac{4}{3}}} \, dx = \frac{\sqrt{1 + 9x^{\frac{4}{3}}}}{3x^{\frac{2}{3}}} \, dx = \frac{\sqrt{1 + 9x\sqrt{x}}}{3x^2} \, dx \]
And so:
\[
\text{Area} = 2\pi \int_a^b r \, ds = 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2} \, dx = 2\pi \int_1^3 \frac{\sqrt{1 + 9x^3}}{3\sqrt{x^2}} \, dx
\]
\[
\text{Area} = \frac{2\pi}{3} \int_1^3 \frac{\sqrt{1 + 9x^3}}{\sqrt{x}} \, dx = \frac{2\pi}{3} \int_1^3 \frac{\sqrt{1 + 9x^3}}{\sqrt{x}} \, dx
\]
\[
\text{or} \quad \frac{2\pi}{3} \int_1^3 \frac{\sqrt{x^2} \cdot \sqrt{1 + 9x^3}}{x} \, dx \approx 378.134593238718249979024
\]

Example 9. Find the area swept out when the graph of the curve defined by parametric equations
\[
\begin{align*}
  x &= \ln t \\
  y &= \sin t
\end{align*}
\]
over the interval \( 1 \leq t \leq \pi \) is revolved about the \( x \)-axis.

Note the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \( ds \) above \( x(t) \) is swept around) is equal to \( y(t) = \sin t \), and so the area of the strip is
\[
\text{Area}_{\text{strip}} \approx 2\pi r \, ds = 2\pi \sin t \, ds.
\]

Recall that \( ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \).

So:
\[
\begin{align*}
  \begin{cases}
    x = \ln t \\
    y = \sin t
  \end{cases} & \Rightarrow 
  \begin{cases}
    \frac{dx}{dt} = \frac{1}{t} \\
    \frac{dy}{dt} = \cos t
  \end{cases}
\end{align*}
\]

And so:
\[
ds = \sqrt{\left(\frac{1}{t}\right)^2 + \left(\cos t\right)^2} \, dt = \sqrt{\frac{1}{t^2} + \cos^2 t} \, dt = \sqrt{\frac{1 + t^2 \cos^2 t}{t^2}} \, dt = \frac{\sqrt{1 + t^2 \cos^2 t}}{t} \, dt
\]

And so, summing up all these area strips from \( t = 1 \) up to \( t = \pi \), we obtain our area integral:
\[
\text{Area} = 2\pi \int_{t=1}^{t=\pi} r \cdot ds = 2\pi \int_1^\pi \sin t \cdot \frac{\sqrt{1 + t^2 \cos^2 t}}{t} \, dt = 2\pi \int_1^\pi \frac{\sin t \sqrt{1 + t^2 \cos^2 t}}{t} \, dt \approx 7.5154961399713
\]
Revolutions About Other Lines

So far, all of our rotations have been about the \( x \)-axis. But there are many other lines out there! We’ll restrict our revolutions to be about horizontal and vertical lines. We’ll just set up the area integrals without solving these integrals (which is good, since they aren’t solvable!). Numerical approximations are also given, for readers with access to calculators or software capable of numerically evaluating definite integrals. For the sake of our precious sanity, we’ll only revolve about horizontal and vertical lines! 😊 As usual, additional worked out examples can be found in the homework solutions.

Example 10. Find the area swept out when the graph of 
\[
y = e^x,
\]
between \( x = 0 \) and \( x = \ln 3 \), is revolved about the \( y \)-axis.

To the left is a graph showing this revolution of \( y = e^x \).
Notice the “strip” whose area is to become our integrand.
The radius of the circular strip (which the little piece of curve \( ds \) above \( x \) is swept around) equals \( x \), and so the area of the strip is
\[
\text{Area}_\text{strip} \approx 2\pi r \cdot ds = 2\pi x \cdot ds.
\]

Recall that
\[
ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + \left(f'(x)\right)^2} \, dx.
\]
Thus:
\[
f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x \quad \Rightarrow \quad ds = \sqrt{1 + \left(e^x\right)^2} \, dx = \sqrt{1 + e^{2x}} \, dx.
\]

And so, summing up all these area strips from \( x = 0 \) up to \( x = \ln 3 \), we obtain our area integral:

\[
\text{Area} = 2\pi \int_a^b r \, ds = 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2} \, dx = 2\pi \int_0^{\ln 3} x \sqrt{1 + e^{2x}} \, dx \approx 9.024262077079845
\]
Example 11. Find the area swept out when the graph of the curve defined by the parametric equations
\[
\begin{align*}
x &= e^t \cos \left(t^2\right) \\
y &= e^t \sin \left(t^2\right)
\end{align*}
\]
over the interval \(0 \leq t \leq \sqrt{\frac{\pi}{2}}\) is revolved about the \(y\)-axis.

Note the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \(ds\) above \(x(t)\) is swept around) is equal to \(x(t) = e^t \cos \left(t^2\right)\), and so the area of the strip is
\[
\text{Area}_{\text{strip}} \approx 2\pi r \ ds = 2\pi e^t \cos \left(t^2\right) \ ds.
\]

Recall that
\[
ds = \sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2} \ dt.
\]
So:
\[
\begin{align*}
x &= e^t \cos \left(t^2\right) \\
y &= e^t \sin \left(t^2\right)
\end{align*}
\]
\[
\begin{align*}
\frac{dx}{dt} &= e^t \cos \left(t^2\right) - 2t e^t \sin \left(t^2\right) \\
\frac{dy}{dt} &= e^t \sin \left(t^2\right) + 2t e^t \cos \left(t^2\right)
\end{align*}
\]
And so:
\[
ds = \sqrt{\left(e^t \cos \left(t^2\right) - 2t e^t \sin \left(t^2\right)\right)^2 + \left(e^t \sin \left(t^2\right) + 2t e^t \cos \left(t^2\right)\right)^2} \ dt
\]

Substituting \(c = \cos \left(t^2\right)\) and \(s = \sin \left(t^2\right)\) will make the subsequent algebra manageable:
\[
ds = e^t \sqrt{\left(c - 2ts\right)^2 + \left(s + 2tc\right)^2} \ dt
\]
\[
= e^t \sqrt{c^2 - 4tsc + 4t^2s^2 + s^2 + 4tsc + 4t^2c^2} \ dt
\]
\[
= e^t \sqrt{c^2 + s^2 + 4t^2\left(c^2 + s^2\right)} \ dt = e^t \sqrt{1 + 4t^2} \ dt
\]

Summing up all these area strips from \(t = 0\) up to \(t = \sqrt{\frac{\pi}{2}}\), we obtain our area integral:
\[
\text{Area} = 2\pi \int_{t=0}^{\sqrt{\frac{\pi}{2}}} r \cdot ds = 2\pi \int_{0}^{\sqrt{\frac{\pi}{2}}} e^t \cos \left(t^2\right) \cdot e^t \sqrt{1 + 4t^2} \ dt = 2\pi \int_{0}^{\sqrt{\frac{\pi}{2}}} e^{2t} \cos \left(t^2\right) \sqrt{1 + 4t^2} \ dt \approx 37.1646170026
\]
Example 12. Find the area swept out when the piece of ellipse defined by parametric equations
\[
\begin{align*}
  x &= 2 \cos t \\
  y &= 3 \sin t
\end{align*}
\]
over the interval \(0 \leq t \leq \frac{\pi}{2}\) is revolved about the line \(x = 4\).

Note the “strip” whose area is to become our integrand. The radius of the circular strip (which the little piece of curve \(ds\) above \(x(t)\) is swept around) is equal to \(r = 4 - x(t) = 4 - 2 \cos t\), and so the area of the strip is
\[
\text{Area}_{\text{strip}} \approx 2\pi r \ ds = 2\pi (4 - 2 \cos t) \ ds = 4\pi (2 - \cos t) \ ds.
\]

Recall that \(ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt\). So:
\[
\begin{align*}
  \frac{dx}{dt} &= -2 \sin t \\
  \frac{dy}{dt} &= 3 \cos t
\end{align*}
\]
And so:
\[
\begin{align*}
  ds &= \sqrt{(2\sin t)^2 + (3\cos t)^2} \ dt = \sqrt{4\sin^2 t + 9\cos^2 t} \ dt \\
  &= \sqrt{4\sin^2 t + 4\cos^2 t + 5\cos^2 t} \ dt = \sqrt{4(\sin^2 t + \cos^2 t) + 5\cos^2 t} \ dt \\
  &= \sqrt{4(1) + 5\cos^2 t} \ dt = \sqrt{4 + 5\cos^2 t} \ dt
\end{align*}
\]

Summing up all these area strips from \(t = 0\) up to \(t = \frac{\pi}{2}\), we obtain our area integral:
\[
\begin{align*}
  \text{Area} &= 2\pi \int_{0}^{\frac{\pi}{2}} r \cdot ds = 4\pi \int_{0}^{\frac{\pi}{2}} (2 - \cos t) \cdot \sqrt{4 + 5\cos^2 t} \ dt \\
  &\approx 33.836436326919943598961
\end{align*}
\]