1.1 Definitions and Terminology

differential equation: An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables.

Example: \( \frac{dy}{dx} = 6y^{2/3} \)

*How can you find y?*

*Solution in implicit form:

*Solution in explicit form:

*Verify your solution.

Differential equations can be classified by type, order, and linearity.

**Type**

**Ordinary** differential equations contain only ordinary derivatives of one or more dependent variables with respect to a single independent variable.

\[
\frac{dy}{dx} = 6y^{2/3}
\]

Examples:

\[
\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 3x = 0
\]

\[
\frac{dx}{ds} + \frac{dy}{ds} + y = 4x
\]
Partial differential equations involve the partial derivatives of one or more dependent variables of two or more independent variables.

\[
\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0
\]

Examples:
\[
\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 1
\]

*Notation: Leibniz

Prime

Newton’s dot

Subscript

Order:
The order of a differential equation (ODE or PDE) is the order of the highest derivative in the equation.

\[
\frac{dy}{dx} = 6 y^{2/3}
\]

*Examples:
\[
\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 3x = 0
\]
\[
\frac{dx}{ds} + \frac{dy}{ds} + y = 4x
\]

First-order ordinary differential equations can be written in differential form.
Example: \((x + y)dx + x^2 dy = 0\)

*If \(y\) is the dependent variable, we can rewrite the equation using Leibniz notation:

*Or prime notation:

An \(n\)th order ODE in one dependent variable can be written in general form as \(F(x, y, y', y'', ..., y^{(n)}) = 0\), where \(F\) is a real-valued function of \(n + 2\) variables.
*Write the previous example in general form.

In this course, we will make the assumption that we can solve $F(x, y, y', y'',..., y^{(n)}) = 0$ uniquely for the highest derivative, $y^{(n)}$, in terms of the remaining $n + 1$ variables. So we will also use the normal form, $\frac{d^n y}{dx^n} = f(x, y, y', ..., y^{(n-1)})$. Here, $f$ is a continuous real-valued function.

*Write the previous example in normal form.

**Linearity:**
An $n^{th}$ order ODE is linear if $F$ is linear in $y, y',..., y^{(n)}$. So an $n^{th}$ order linear ODE can be written as $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ where

1. each coefficient, $a_i(x)$, depends at most on the independent variable $x$, and
2. the dependent variable and all its derivatives are of the first degree.

Example: $xdy + (y + xy - xe^x)dx = 0$

*Is this ODE linear in $y$?

*Is this ODE linear in $x$?

**Solution** of an ODE: any real-valued function, $\phi$, defined on an interval $I$ and possessing at least $n$ derivatives that are continuous on $I$, which when substituted into an $n^{th}$ order ODE reduces the equation to an identity.
$I$ is called the interval of definition, interval of existence, interval of validity, or domain.

*Example: Verify that $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ is a solution of $\frac{dy}{dt} + 20y = 24$ on $I = (-\infty, \infty)$. 
Example: Verify that $y = xe^x$ is a solution of $y'' - 2y' + y = 0$ on $I = (-\infty, \infty)$.

Notice that $y = 0$ is also a solution of the previous example. A solution that is identically zero on $I$ is called a **trivial solution**.

*Verify that $y = 0$ is a solution of $y'' - 2y' + y = 0$ on $I = (-\infty, \infty)$.

The domain of the function $\phi(x)$ may not be the same as the interval of definition of the solution $\phi(x)$.

Example: Find the **general solution** of $2y' = y^3 \cos x$

*Implicit form:

We solved $F(x, y, y') = 0$ and found a set $G(x, y, c) = 0$, called a **one-parameter family of solutions**.

*One-parameter family of solutions:

For an nth order ODE $F(x, y, y', y'', \ldots, y^{(n)}) = 0$, we look for an **n-parameter family of solutions**, $G(x, y, c_1, c_2, \ldots, c_n) = 0$.

*Explicit form:
*Use these initial conditions to find a **particular solution** (free of arbitrary parameters):

\[ y(0) = 1 \]

* \[ \phi(x) = \]

*What is the domain of the function \( \phi(x) \)?

*What is an interval of definition of the solution \( \phi(x) \)?

We usually state the largest interval as the interval of definition.

*Is \( y = 0 \) a trivial solution?

Notice that the solution \( y = 0 \) is not achieved by assigning a value to \( c \). This makes \( y = 0 \) a **singular solution**.

**Piecewise-defined solutions**

*Example: Solve \( xy' - 4y = 0 \), \( y(1) = 1 \), \( y(-1) = -1 \).