1.2 Initial Value Problems

An initial value problem involves solving a differential equation subject to certain conditions. We may wish to solve an nth order IVP on an interval containing $x_0$:

Solve $\frac{d^n y}{dx^n} = f(x, y, y',..., y^{(n-1)})$ subject to $y(x_0) = y_o, y'(x_0) = y_1,\ldots, y^{(n-1)}(x_0) = y_{n-1}$.

*Geometrically, solving a first order IVP means

*Solving a second order IVP means

etc.

*Example: Solve $y' - y = 0$ subject to $y(1) = 5e$.

*Example: $y = c_1x + c_2x^3$ is a two-parameter family of solutions of $y - xy' + \frac{1}{3}x^2y'' = 0$.

*Find a particular solution subject to $y(1) = 6$ and $y'(1) = 2$.

When solving an IVP there are two concerns:
Does a solution exist?
If a solution exists, is it unique?
**Theorem:**
Let $R$ be a rectangular region in the $xy$-plane defined by $a < x < b$, $c < y < d$, that contains $(x_0, y_0)$ in the interior.

If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on $R$, then there exists some interval $I_0$: $x_0 - h < x < x_0 + h$, $h > 0$ contained in $a < x < b$ and a unique function $y(x)$ defined on $I_0$ that is the solution of the first order IVP.

*Example:
\[
\frac{dy}{dx} - y = x
\]

\[f(x, y) = \]

\[\frac{\partial f}{\partial y} = \]

*Example:
\[
(y - x)\frac{dy}{dx} = y + x
\]

\[f(x, y) = \]

\[\frac{\partial f}{\partial y} = \]

*Example: Solve $y \frac{dy}{dx} = 3x$ subject to $y(2) = -4$.
First, does a unique solution exist through $(2, -4)$?

\[f(x, y) = \]

\[\frac{\partial f}{\partial y} = \]
The conditions of the theorem are sufficient, but not necessary.

*What particular solution do we get for these initial conditions?*

*y(0) = 0*  

*y(4) = 0*