Math 6D, Ch. 2
First-Order Differential Equations

2.1 Solution curves without the solution

Notice that Theorem 1.1 may tell us that a solution exists, but does not give information about how to solve. In fact, some DE’s are not solvable by equation-specific methods. We may need to get information about a solution without actually solving the DE.

Idea: We want to get information about \( y \) from \( \frac{dy}{dx} \).

* \( \frac{dy}{dx} \) gives us information about the ___________ of \( y \).

* A solution, \( y = y(x) \), of a first-order DE \( \frac{dy}{dx} = f(x, y) \) is necessarily ________________ on its interval of definition, so it is also ________________.

* So it has no __________ and must have a ___________ ___________ at each point.

Pick a point \( (x, y) \) in the region over which \( f \) is defined. The value \( f(x, y) \) assigns to that point is

* the ___________ of a line segment called a **lineal element**.

Example: \( \frac{dy}{dx} = 0.1xy \)

* At the point \((4, 5)\), the slope of the lineal element is \( f(4, 5) = \) ____________.
If a solution curve passes through \((4, 5)\) it does so tangent to this line segment.

Picture:

* Are we guaranteed to have a unique solution to \( \frac{dy}{dx} = 0.1xy \), \( y(4) = 5 \)?

If we evaluate \( f(x, y) \) over a rectangular grid and draw the lineal elements then we get a **direction field** or a **slope field**.
View a slope field using the TI-89:
Mode: Diff Equations
\[ t0 = 4 \]
\[ y1' = 0.1 \times t \times y1 \]
\[ y1 = 5 \]

* Solve \( \frac{dy}{dx} = 0.1xy \), \( y(4) = 5 \).

Some reminders from Calculus I:
We can get information from the sign of \( \frac{dy}{dx} \).

* If \( \frac{dy}{dx} > 0 \) on I \( \Rightarrow \) y is ____________ on I.

* If \( \frac{dy}{dx} < 0 \) on I \( \Rightarrow \) y is ____________ on I.

We can also get information by determining where \( \frac{dy}{dx} = 0 \).

Qualitative Analysis of Autonomous Differential Equations

Autonomous first-order DE’s are those in which the independent variable does not appear explicitly.

Example: Which of the following DE’s are autonomous?

\[ \frac{dy}{dx} = y - y^3 \]
\[ \frac{dy}{dx} = 0.1xy \]
**Critical points**

A real number is a **critical point** of an autonomous DE if it is a zero of \( \frac{dy}{dx} = f(y) \). These points are also called **equilibrium points** or **stationary points**.

If \( c \) is a critical point of \( \frac{dy}{dx} = f(y) \), then \( y(x) = c \) is a constant solution of the autonomous equation.

Zero of \( f \) \( \Rightarrow f(c) = 0 \) \( \Rightarrow y(x) = c \)

*Verify that \( y(x) = c \) is a solution of \( \frac{dy}{dx} = f(y) \).

So if \( c \) is a critical point of \( \frac{dy}{dx} = f(y) \) then \( y(x) = c \) is a **constant solution** or **equilibrium solution**.

We can determine where a **nonconstant solution** is increasing or decreasing by looking at the sign of \( \frac{dy}{dx} \).

Example: \( \frac{dy}{dx} = y^2 - y^3 \)

*Find the critical points.

*Put these critical values on a **phase line** and determine the **one-dimensional phase portrait**.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of ( f(y) )</th>
<th>( F(y) )</th>
<th>arrow</th>
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*Are we guaranteed of a unique solution to this DE at every point in the plane?

The critical points divide the plane into subregions, $R_i$. Notice that a nonconstant solution cannot cross the graph of a constant solution if we are guaranteed to have a unique solution for each point in the plane.

*Translate the phase portrait to the plane.

Note: Within a subregion there are no relative extrema, and there cannot be oscillation. Also, as $x \to \pm \infty$, $y(x) \to$ critical point, if $y(x)$ is bounded.

We could also look for clues about concavity by analyzing the second derivative.

*For $\frac{dy}{dx} = y^2 - y^3$, find $\frac{d^2y}{dx^2}$.

*Find the critical points of the second derivative.
*Put the critical points on a phase line and determine concavity.

These results should agree with the solution curves you drew in the plane.

*Is $\frac{dy}{dx} = y^2 - y^3$ separable? How could it be solved analytically?

A critical point can be classified as an **attractor** (asymptotically stable), a **repeller** (unstable), or **semi-stable**.

An **attractor** has both arrows pointing toward it on the phase portrait.
A **repeller** has both arrows pointing away from it on the phase portrait.
A critical point is **semi-stable** if it attracts from one side, and repels from the other.

*For $\frac{dy}{dx} = y^2 - y^3$, classify each critical point.

*Example: $\frac{dy}{dx} = y^3 - 6y^2 + 8y$. Find the critical point(s) and phase portrait. Classify the critical point(s). Sketch typical solution curves in the xy-plane.
*Example: \( \frac{dy}{dx} = (y + 3)^2 \). Find the critical point(s) and phase portrait. Classify the critical point(s). Sketch typical solution curves in the xy-plane.

*Solve \( \frac{dy}{dx} = (y + 3)^2 \). Write your solution in explicit form.

*Solve \( \frac{dy}{dx} = (y + 3)^2 \), \( y(0) = -2 \).

*Solve \( \frac{dy}{dx} = (y + 3)^2 \), \( y(0) = -4 \).